NONABELIAN GENERALIZED LAX PAIRS, THE CLASSICAL YANG-BAXTER EQUATION AND POSTLIE ALGEBRAS

XIANG NI, CHENGMING BAI, AND LI GUO

ABSTRACT. We generalize the classical study of (generalized) Lax pairs and the related $\mathcal{O}$-operators and the (modified) classical Yang-Baxter equation by introducing the concepts of nonabelian generalized Lax pairs, extended $\mathcal{O}$-operators and the extended classical Yang-Baxter equation. We study in this context the nonabelian generalized r-matrix ansatz and the related double Lie algebra structures. Relationship between extended $\mathcal{O}$-operators and the extended classical Yang-Baxter equation is established, especially for self-dual Lie algebras. This relationship allows us to obtain explicit description of the Manin triples for a new class of Lie bialgebras. Furthermore, we show that a natural structure of PostLie algebra is behind $\mathcal{O}$-operators and fits in a setup of triple Lie algebra that produces self-dual nonabelian generalized Lax pairs.

CONTENTS

Key words and phrases. Lax pair, Lie algebra, Lie bialgebra, $\mathcal{O}$-operator, classical Yang-Baxter equation, PostLie algebra.

1. Introduction

This paper is devoted to a systematic study of the integrable Hamiltonian systems and the related (generalized) classical Yang-Baxter equation (CYBE) in a broad context that generalizes or extends the studies of Bordemann [10], Hodge and Yakimov [25], Kosmann-Schwarzbach and Magri [29], and Semonov-Tian-Shansky [43].

Since their introduction by Lax in 1968, Lax pairs have become important in giving conservation laws in an integrable system. In connection with r-matrices satisfying the classical Yang-Baxter equation (CYBE), Poisson commuting conservation laws could be constructed. Main contributors in this direction include Adler [1], Babelon and Viallet [4, 5], Belavin and Drinfeld [8, 9, 17], Faddeev [21], Kostant [30], Reyman and Semonov-Tian-Shansky [41, 43], Sklyanin [45, 46] and Symes [48, 49].

In [10] Bordemann introduced the notions of generalized Lax pairs and generalized r-matrix ansatz. He achieved this through replacing the well-known Lax equation [32]

$$\frac{dL}{dt} = [L, M]$$

by

$$\frac{dL}{dt} = -\rho(M)L, \tag{1}$$

where ρ is any representation of a Lie algebra $\mathfrak{g}$ in a representation space V, M is a $\mathfrak{g}$-valued function on the phase space and L is a V-valued function on the phase space, reducing to the Lax equation when V is taken to be $\mathfrak{g}$ and ρ is taken to be the adjoint representation. In this generality, the correct framework to extend the classical r-matrices is through their operator forms, later called $\mathcal{O}$-operators by Kupershmidt [31].

The classical Yang-Baxter equation, through its operator form and tensor form, plays a central role in relating several areas in mathematics. For the most part, the operator form is more convenient in application to integrable systems. For example, the modified classical Yang-Baxter equation is solely defined in the operator form. Nevertheless, the tensor form of the CYBE is the classical limit of the quantum Yang-Baxter equation, and its solutions give rise to important concepts such as (coboundary) Lie bialgebras. Thus it is desirable to work with both forms of the CYBE.

In the present paper, we keep both forms of the CYBE in mind while we generalize the previous works. For the operator form, we further generalize the work of Bordemann and Kupershmidt by introducing the concepts of an **$\mathcal{O}$-operator of weight λ** (for a constant λ) and an **extended $\mathcal{O}$-operator**. This is motivated by our attempt to extend generalized Lax pairs of Bordemann to **nonabelian generalized Lax pairs**, by still considering Eq. (1) but replacing the representation space V by any Lie algebra $\mathfrak{a}$ and the representation ρ by any Lie algebra homomorphism from $\mathfrak{g}$ to $\mathrm{Der}(\mathfrak{a})$ consisting of derivations of $\mathfrak{a}$. The setting of Bordemann is recovered when $\mathfrak{a}$ is taken to be an abelian Lie algebra. We extend the generalized r-matrix ansatz of Bordemann to the nonabelian context and show that extended $\mathcal{O}$-operators ensure the consistency of a Lie structure on $\mathfrak{a}^*$ defined by the r-matrices. For the tensor form, we introduce the concept of the **extended classical Yang-Baxter equation** and establish their relationship with extended $\mathcal{O}$-operators as in the case of (the tensor form and operator form) of the CYBE. We further extend the well-known work of Drinfeld on quasitriangular Lie bialgebras from the CYBE to what we dubbed **type II quasitriangular Lie bialgebras** from a case of the extended classical Yang-Baxter equation, called

the **type II CYBE**. The corresponding Drinfeld's doubles and Manin triples are studied carefully as in the classical case by Hodge and Yakimov [25], for their importance in the classification of the Poisson homogeneous spaces and symplectic leaves of the corresponding Poisson-Lie groups [18, 25, 44, 52].

As it turns out, an $\mathcal{O}$-operator of weight λ is related to the concept of a PostLie algebra that has recently arisen from the quite different context of operads [51]. More precisely, an $\mathcal{O}$-operator, paired with a $\mathfrak{g}$-Lie algebra, gives a PostLie algebra. In particular, Baxter Lie algebras and quasitriangular Lie bialgebras give rise to PostLie algebras. Furthermore the well-known relation [12] between pre-Lie algebras and dendriform dialgebras, in connection with the classical relation between associative algebras and Lie algebras, can be extended to that between PostLie algebras and dendriform trialgebras. Quite unexpectedly, this "digression" of $\mathcal{O}$-operators to PostLie algebra is tired up with our primary application of $\mathcal{O}$-operators in studying nonabelian generalized Lax pairs: We introduce the concept of a **triple Lie algebra** to construct self-dual nonabelian generalized Lax pairs and show that a natural example of a triple Lie algebra is provided by the PostLie algebra from a Rota-Baxter operator action on a complex simple Lie algebra.

We next give a summary of this paper.

We begin our study by introducing the concept of a nonabelian generalized Lax pair. We write down a "nonabelian generalized r-matrix ansatz" to produce Poisson commuting conservation laws. The idea is to use the Lie-Poisson structure on the representation space (equipped with a Lie bracket) to twist the "generalized r-matrix ansatz" of Bordemann [10]. In geometry, this construction might be understood as "twisting" a Hamiltonian system (Poisson bracket) by the Hamiltonian system (Lie-Poisson bracket) on the dual space of a Lie algebra. The notions $\mathcal{O}$-operator of weight λ and extended $\mathcal{O}$-operator of weight λ with extension β of mass (ν,κ,μ) (for constants (ν,κ,μ)) appear naturally when we investigate sufficient conditions for the double Lie algebra structures needed for the existence of the ansatz.

To generalize the well-known relationship between the operator form and tensor form of the CYBE, we introduce in Section 3 the concept of an extended CYBE and relate it to extended $\mathcal{O}$-operators. Applications to Lie bialgebras are given. In particular, we study in detail the structure of the Manin triple of a type II quasitriangular Lie bialgebra.

In Section 4, we study the case of self-dual Lie algebras. The ideal is to use a nondegenerate symmetric and invariant bilinear form of a self-dual Lie algebra to identify the adjoint representation and coadjoint representation [43]. Some new aspects on Lie bialgebras are given along this approach, for example, new examples of (type II) factorizable quasitriangular Lie bialgebras are provided.

We show in Section 5 that there naturally exists an algebraic structure behind an $\mathcal{O}$-operator of weight λ, namely, the PostLie algebra discovered in a study of operads [51]. We also reveal a relation between PostLie algebras and dendriform trialgebras of Loday and Ronco [36] by a commutative diagram.

In Section 6, we provide a framework of triple Lie algebras to construct a class of nonabelian generalized Lax pairs for which the corresponding r-matrix ansatz can be written down explicitly [13]. We show that PostLie algebras provide natural instances of such triple Lie algebras.

Finally in an appendix, we give a geometric explanation of extended $\mathcal{O}$-operators.

Conventions: In this paper, the base field is taken to be $\mathbb{R}$ of real numbers unless otherwise specified. This is the field from which we take all the constants and over which we take all the associative and Lie algebras, vector spaces, linear maps and tensor products, etc. All Lie algebras, vector spaces and manifolds are assumed to be finite-dimensional, although many results still hold in infinite-dimensional case.

Acknowledgements: The second author was supported in part by the National Natural Science Foundation of China (1062 1101), NKBRPC (2006CB805905) and SRFDP (200800550015). The third author was supported by NSF grant DMS 0505445 and thanks the Chern Institute of Mathematics at Nankai University for hospitality.

2. Nonabelian generalized Lax pairs and extended $\mathcal{O}$-operators

We begin with generalizing the generalized Lax pairs of Bordemann [10] further to nonabelian generalized Lax pairs. By studying generalized r-matrix ansatz and double Lie algebra structures in this context, we are motivated to introducing the concept of an extended $\mathcal{O}$-operator, generalizing the work of Bordemann and Kupershmidt [31] in several directions. The case of adjoint representations is studied separately.

2.1. **Nonabelian generalized Lax pairs.** We first introduce a suitable replacement of Lie algebra representations in order to extend generalized Lax pairs to the nonabelian context.

Definition 2.1. (i) Let $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$, or simply $\mathfrak{g}$, denote a Lie algebra $\mathfrak{g}$ with Lie bracket $[\ ,\ ]_{\mathfrak{g}}$.

(ii) For a Lie algebra $\mathfrak{b}$, let $\mathrm{Der}_{\mathbb{R}}\mathfrak{b}$ denote the Lie algebra of derivations of $\mathfrak{b}$.

(iii) Let $\mathfrak{a}$ be a Lie algebra. An $\mathfrak{a}$-**Lie algebra** is a triple $(\mathfrak{b}, [\ ,\ ]_{\mathfrak{b}}, \pi)$ consisting of a Lie algebra $(\mathfrak{b}, [\ ,\ ]_{\mathfrak{b}})$ and a Lie algebra homomorphism $\pi : \mathfrak{a} \to \mathrm{Der}_{\mathbb{R}}\mathfrak{b}$. To simplify the notation, we also let $(\mathfrak{b}, \pi)$ or simply $\mathfrak{b}$ denote $(\mathfrak{b}, [\ ,\ ]_{\mathfrak{b}}, \pi)$.

(iv) Let $\mathfrak{a}$ be a Lie algebra and let $(\mathfrak{g}, \pi)$ be an $\mathfrak{a}$-Lie algebra. Let $a \cdot b$ denote $\pi(a)b$ for $a \in \mathfrak{a}$ and $b \in \mathfrak{g}$.

According to [26], if $(\mathfrak{b}, \pi)$ is an $\mathfrak{a}$-Lie algebra, then there exists a unique Lie algebra structure on the direct sum $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ of the underlying vector spaces $\mathfrak{a}$ and $\mathfrak{b}$ such that $\mathfrak{a}$ and $\mathfrak{b}$ are subalgebras and $[x, y] = \pi(x)y$ for $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$. Further, $\mathfrak{a}$ is a subalgebra and $\mathfrak{b}$ is an ideal of the Lie algebra $\mathfrak{g}$.

Let (P, w) be a Poisson manifold with the Poisson bivector $w \in \bigwedge^2 T(M)$ which induces a Poisson bracket $\{,\}$ on $C^\infty(P)$. A smooth function f on P, which is called an **observable**, determines a **Hamiltonian vector field** X_f by

$$X_f\, g \equiv \{f, g\}, \quad g \in C^\infty(P).$$

If a Hamiltonian system is modeled by a Poisson manifold (P, w) (the phase space of the system) and a Hamiltonian $\mathcal{H} \in C^\infty(P)$, its time-evolution is given by the following integral curves of the Hamiltonian vector field $X_{\mathcal{H}}$ on P corresponding to $\mathcal{H}$:

$$X_{\mathcal{H}}(f) \equiv \{\mathcal{H}, f\}, \quad \forall f \in C^\infty(P).$$

It follows that

$$\frac{df}{dt} = \{\mathcal{H}, f\},$$

in the sense that $(d/dt)(f(m(t))) = \{\mathcal{H}, f\}(m(t))$ for an integral curve $m(t)$ of $X_{\mathcal{H}}$. As usual, an observable f is called a **conservation law** or **conserved** if $\{\mathcal{H}, f\} = 0$. Two conservation laws f_1, f_2 on a Poisson manifold are **in involution** or **Poisson commuting** if $\{f_1, f_2\} = 0$. Moreover, a Hamiltonian system $(P, w, \mathcal{H})$ is called **completely integrable** if it has the maximum number of conserved observables in involution [13].

An important procedure to obtain Poisson commuting observables and completely integrable Hamiltonian systems is through the concept of **Lax pairs** [32] which was generalized by Bordemann [10] to **generalized Lax pairs**. We now generalize this further to the following concept.

Definition 2.2. (i) A **nonabelian generalized Lax pair** for a Hamiltonian system $(P, w, \mathcal{H})$ is a quintuple $(\mathfrak{g}, \rho, \mathfrak{a}, L, M)$ satisfying the following conditions:

(a) $\mathfrak{g}$ is a (finite-dimensional) Lie algebra;
(b) $(\mathfrak{a}, \rho)$ is a (finite-dimensional) $\mathfrak{g}$-Lie algebra with the Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathrm{Der}_{\mathbb{R}}(\mathfrak{a})$;
(c) $L : P \to \mathfrak{a}$ is a smooth map,
(d) $M : P \to \mathfrak{g}$ is a smooth map such that

$$dL(p)X_{\mathcal{H}}(p) = -\rho(M(p))L(p), \quad \forall p \in P. \tag{2}$$

(ii) A nonabelian generalized Lax pair $(\mathfrak{g}, \rho, \mathfrak{a}, L, M)$ is said to be **self-dual** if $\mathfrak{a}$ is equipped with a nondegenerate symmetric bilinear form $\mathfrak{B} : \mathfrak{a} \otimes \mathfrak{a} \to \mathbb{R}$ such that

$$\mathfrak{B}([x, y]_{\mathfrak{a}}, z) = \mathfrak{B}(x, [y, z]_{\mathfrak{a}}), \quad \forall x, y, z \in \mathfrak{a}, \tag{3}$$

$$\mathfrak{B}(\rho(\xi)x, y) + \mathfrak{B}(x, \rho(\xi)y) = 0, \quad \forall \xi \in \mathfrak{g}, x, y \in \mathfrak{a}. \tag{4}$$

Note that a bilinear form on a Lie algebra satisfying Eq. (3) is called **invariant** and a Lie algebra endowed with a nondegenerate symmetric invariant bilinear form is called a **self-dual Lie algebra** [22].

By the chain rule, Eq. (2) is equivalent to

$$\frac{dL}{dt} = -\rho(M)L. \tag{5}$$

Remark 2.3. (i) When the Lie bracket on $\mathfrak{a}$ happens to be trivial, the $\mathfrak{g}$-Lie algebra $(\mathfrak{a}, \rho)$ becomes a representation of $\mathfrak{g}$ and the nonabelian generalized Lax pair becomes the **generalized Lax pair** in the sense of Bordemann [10].

(ii) For $\mathfrak{a} = \mathfrak{g}$ and $\rho = \mathrm{ad}$, Eq. (5) is the usual Lax equation. Moreover, the Lax pair can be realized as a nonabelian generalized Lax pair in two different ways, by either taking ρ to be ad and $\mathfrak{a}$ to be the Lie algebra $\mathfrak{g}$, or taking ρ to be ad and $\mathfrak{a}$ to be the underlying vector space of $\mathfrak{g}$ equipped with the trivial Lie bracket.

Let G be a connected Lie group whose Lie algebra is $\mathfrak{g}$ such that ρ exponentiates to a representation of G in V which we shall also call ρ. We first show that, as in the case of Lax pairs and generalized Lax pairs [10], nonabelian generalized Lax pairs also give conservation laws.

Proposition 2.4. *Let $(\mathfrak{g}, \rho, \mathfrak{a}, L, M)$ be a nonabelian generalized Lax pair for a Hamiltonian system $(P, w, \mathcal{H})$. If $f : \mathfrak{a} \to \mathbb{R}$ is a G-invariant smooth function, i.e., $f(\rho(g)x) = f(x)$ for all $g \in G$ and $x \in \mathfrak{a}$, then $f \circ L$ is a conservation law, i.e.,*

$$\frac{d(f \circ L)}{dt} = \{f \circ L, \mathcal{H}\} = 0.$$

Proof. Since G-invariant functions are always constant on each G-orbit, we have

$$df(x)\rho(\xi)x = 0, \quad \forall \xi \in \mathfrak{g}, x \in \mathfrak{a}.$$

So

$$\frac{d}{dt}(f \circ L) = df(L)\frac{d}{dt}L = -df(L)\rho(M)L = 0.$$

$\square$

Let $\{e_i\}_{1 \le i \le \dim \mathfrak{a}}$ be a basis of $\mathfrak{a}$ and $\{T_A\}_{1 \le A \le \dim \mathfrak{g}}$ be a basis of $\mathfrak{g}$. For any $x = \sum_i x^i e_i \in \mathfrak{a}$ and $\xi = \sum_A \xi^A T_A \in \mathfrak{g}$, we set $(\rho(\xi)x)^i = \sum_{A,j} \xi^A x^j \rho^i_{Aj}$. On the other hand, suppose that the Lie algebra

structure on $\mathfrak{a}$ is given by $[e_i, e_j]_{\mathfrak{a}} = \sum\limits_k c_{ij}^k e_k$. The Poisson bracket $\{f \circ L, h \circ L\}$ for arbitrary smooth functions $f, h : \mathfrak{a} \to \mathbb{R}$ is

$$\{f \circ L, h \circ L\} = \sum_{i,j} \frac{\partial f}{\partial x^i} \circ L \frac{\partial h}{\partial x^j} \circ L\{L^i, L^j\}. \tag{6}$$

Now consider smooth maps which we shall call **classical r-matrices** (following [10])

$$r_+, r_- : \mathfrak{a} \times P \to \mathfrak{a} \otimes \mathfrak{g}$$

and make the following **nonabelian generalized r-matrix ansatz**:

$$\{L^i, L^j\}(p) = -\sum_{A,k} r_+^{iA}(L(p), p)\rho_{Ak}^j L^k(p) + \sum_{A,k} r_-^{jA}(L(p), p)\rho_{Ak}^i L^k(p) - \sum_k \theta_i(p) c_{ik}^j L^k(p), \tag{7}$$

where $\theta : P \to \mathfrak{a}$ is a smooth function and $\theta_i = x^i \circ \theta : P \to \mathbb{R}$, $1 \le i \le \dim\mathfrak{a}$.

When $\theta = 0$, the third term on the right hand side vanishes and the ansatz is reduced to Bordemann's **generalized r-matrix ansatz** [10]. Generalizing the work of Bordemann, we next show that the nonabelian generalized r-matrix ansatz gives Poisson commuting conservation laws.

Proposition 2.5. *Let $(\mathfrak{g}, \rho, \mathfrak{a}, L, M)$ be a nonabelian generalized Lax pair for a Hamiltonian system $(P, w, \mathcal{H})$ allowing for classical r-matrices that obey Eq. (7). Then for two real-valued G-invariant and $\mathrm{Ad}_{\mathfrak{a}}$-invariant functions f and h on $\mathfrak{a}$, we have $\{f \circ L, h \circ L\} = 0$.*

Proof. Using Eq. (6), we get

$$\begin{aligned}\{f \circ L, h \circ L\} &= -\sum_{i,A,j,k} \frac{\partial f}{\partial x^i} \circ L r_+^{iA} \underline{\frac{\partial h}{\partial x^j} \circ L \rho_{Ak}^j L^k} + \sum_{i,A,j,k} \underline{\frac{\partial f}{\partial x^i} \circ L \rho_{Ak}^i L^k} \frac{\partial h}{\partial x^j} \circ L r_-^{jA} \\ &\quad - \sum_{i,j,k} \underline{\frac{\partial h}{\partial x^j} \circ L c_{ik}^j L^k} \frac{\partial f}{\partial x^i} \circ L \theta_i.\end{aligned}$$

The underlined terms are zero because of infinitesimal G-invariance and $\mathrm{Ad}_{\mathfrak{a}}$-invariance of f and h. ◻

As pointed out by Bordemann in [10], for $\mathfrak{a} = \mathfrak{g}$ (with the trivial bracket) and $\rho = \mathrm{ad}$, the classical r-matrices take values in $\mathfrak{g} \otimes \mathfrak{g}$, and the above conclusion becomes the classical fact [4] that arbitrary trace polynomials of L Poisson commute among themselves.

The Lie bracket conditions on the left hand side of Eq. (7) impose consistence restrains on the classical r-matrices on the right hand side. In the case of constant r-matrices (namely L-independent) that we will consider below, as observed by Bordemann, the space spanned by the component functions L^i behaves like a finite-dimensional Lie subalgebra of the Poisson algebra of functions on the phase space (P, w) since the right-hand side of Eq. (7) is linear in L. Suppose one wants to collectively investigate all nonabelian generalized Lax pairs that are defined on a given Hamiltonian system $(P, w, \mathcal{H})$, that have given $\mathfrak{g}$, ρ and $\mathfrak{a}$, and that satisfy Eq. (7) with given classical r-matrices r_+ and r_-. Then one is led to the following stronger condition than the above mentioned consistence restrains imposed on Eq. (7):

Condition 2.6. The quantities

$$f_k^{ij} \equiv -\sum_A r_+^{iA}\rho_{Ak}^j + \sum_A r_-^{jA}\rho_{Ak}^i - \theta_i c_{ik}^j$$

should be the structure constants of a Lie structure on $\mathfrak{a}^*$.

To obtain an index-free form of Condition 2.6, we first give the following lemma.

Lemma 2.7. *Let $\mathfrak{g}$ be a Lie algebra and $(\mathfrak{a}, \rho)$ be a $\mathfrak{g}$-Lie algebra. Let $\mathfrak{B} : \mathfrak{a} \otimes \mathfrak{a} \to \mathbb{R}$ be a nondegenerate bilinear form on $\mathfrak{a}$ which can be identified as an invertible linear map $\varphi : \mathfrak{a} \to \mathfrak{a}^*$ through*

$$\mathfrak{B}(x, y) = \langle \varphi(x), y \rangle, \quad \forall x, y \in \mathfrak{a}. \tag{8}$$

Let $(\mathfrak{a}^*, \rho_\varphi)$ be the $\mathfrak{g}$-Lie algebra through φ by transporting the $\mathfrak{g}$-Lie algebra structure on $\mathfrak{a}$. More precisely, define the Lie bracket on $\mathfrak{a}^*$ by

$$[a^*, b^*]_{\mathfrak{a}^*} = \varphi([\varphi^{-1}(a^*), \varphi^{-1}(b^*)]_{\mathfrak{a}}), \quad \forall a^*, b^* \in \mathfrak{a}^* \tag{9}$$

and define a linear map

$$\rho_\varphi : \mathfrak{g} \to \mathrm{End}_{\mathbb{R}}(\mathfrak{a}^*), \quad \rho_\varphi(\xi) a^* \equiv \varphi \rho(\xi) \varphi^{-1}(a^*), \quad \forall a^* \in \mathfrak{a}^*, \xi \in \mathfrak{g}. \tag{10}$$

If $\mathfrak{B}$ satisfies Eq. (4), then ρ_φ is just the dual representation ρ^* of ρ which is defined by

$$\langle \rho^*(\xi) a^*, x \rangle = -\langle a^*, \rho(\xi) x \rangle, \quad \forall \xi \in \mathfrak{g}, x \in \mathfrak{a}, a^* \in \mathfrak{a}^*.$$

In this case, $(\mathfrak{a}^*, \rho^*)$ is a $\mathfrak{g}$-Lie algebra with the Lie bracket defined by Eq. (9).

Proof. If $\mathfrak{B}$ satisfies Eq. (4), then for any $\xi \in \mathfrak{g}$, $x, y \in \mathfrak{a}$,

$$\langle \varphi(\rho(\xi) x), y \rangle = -\langle \varphi(x), \rho(\xi) y \rangle \Rightarrow \varphi \rho(\xi) = \rho^*(\xi) \varphi, \quad \forall \xi \in \mathfrak{g}.$$

Hence $\rho_\varphi = \rho^*$. So the conclusion holds. □

Assume that $\mathfrak{a}$ is equipped with a nondegenerate symmetric bilinear form $\mathfrak{B} : \mathfrak{a} \otimes \mathfrak{a} \to \mathbb{R}$ for which the nonabelian generalized Lax pair $(\mathfrak{g}, \rho, \mathfrak{a}, L, M)$ is self-dual. Let $\mathfrak{a}^*$ be equipped with the Lie bracket defined by Eq. (9). By Lemma 2.7, $(\mathfrak{a}^*, [\ ,\ ]_{\mathfrak{a}^*}, \rho^*)$ is a $\mathfrak{g}$-Lie algebra. Since $\mathfrak{B}$ is nondegenerate and symmetric, we can choose a basis $\{e_i\}_{1 \le i \le \dim \mathfrak{a}}$ of $\mathfrak{a}$ such that

$$b_{ij} \equiv \mathfrak{B}(e_i, e_j) = \langle \varphi(e_i), e_j \rangle = 0, \text{ if } i \neq j; \quad b_{ii} \equiv \mathfrak{B}(e_i, e_i) = \langle \varphi(e_i), e_i \rangle \neq 0.$$

Thus, $\varphi(e_i) = b_{ii} e_i^*$, where $\{e_i^*\}_{1 \le i \le \dim \mathfrak{a}}$ is the dual basis of $\{e_i\}_{1 \le i \le \dim \mathfrak{a}}$. Since $\mathfrak{B}([e_i, e_j]_{\mathfrak{a}}, e_k) + \mathfrak{B}(e_j, [e_i, e_k]_{\mathfrak{a}}) = 0$, we have $c_{ij}^k b_{kk} + c_{ik}^j b_{jj} = 0$. Therefore,

$$[e_i^*, e_j^*]_{\mathfrak{a}^*} = \varphi[\varphi^{-1}(e_i^*), \varphi^{-1}(e_j^*)]_{\mathfrak{a}} = \varphi([\frac{e_i}{b_{ii}}, \frac{e_j}{b_{jj}}]_{\mathfrak{a}}) = \frac{\sum\limits_k c_{ij}^k b_{kk} e_k}{b_{ii} b_{jj}} = \frac{-\sum\limits_k c_{ik}^j e_k}{b_{ii}}.$$

Now we set $\theta_i \equiv \frac{\lambda}{b_{ii}}$ for $\lambda \in \mathbb{R}$. On the other hand, since $\mathfrak{a} \otimes \mathfrak{g} \simeq \mathrm{Hom}(\mathfrak{a}^*, \mathfrak{g})$, r_+ and r_- can be considered as linear maps $\mathfrak{a}^* \to \mathfrak{g} : x = x_i e_i^* \to r_\pm(x) \equiv \sum\limits_{i,A} x_i r_\pm^{iA} T_A$. Set

$$\mathfrak{k} \equiv \mathfrak{a}^*, \quad \pi \equiv \rho^*, \quad \xi \cdot x \equiv \pi(\xi) x, \quad x \in \mathfrak{k}, \xi \in \mathfrak{g}.$$

Then Condition 2.6 can be reformulated as follows:

Condition 2.8. (Double Lie algebra structure) The product

$$[x, y]_R \equiv r_+(x) \cdot y - r_-(y) \cdot x + \lambda [x, y]_{\mathfrak{k}}, \quad \forall x, y \in \mathfrak{k}.$$

defines a Lie bracket on $\mathfrak{k}$.

Define

$$r \equiv (r_+ + r_-)/2, \quad \beta \equiv (r_+ - r_-)/2. \tag{11}$$

Then $r_\pm = r \pm \beta$. Moreover, we have the following result:

Proposition 2.9. *Condition 2.8 holds if and only if for any $x, y, z \in \mathfrak{k}$,*

(i) $[x, y]_R = r(x) \cdot y - r(y) \cdot x + \lambda [x, y]_{\mathfrak{k}} \Leftrightarrow \beta(x) \cdot y + \beta(y) \cdot x = 0$,

(ii) $([r(x), r(y)]_{\mathfrak{g}} - r([x, y]_R)) \cdot z + ([r(y), r(z)]_{\mathfrak{g}} - r([y, z]_R)) \cdot x + ([r(z), r(x)]_{\mathfrak{g}} - r([z, x]_R)) \cdot y = 0.$

To simplify notations, for an expression $\eta(x,y,z)$ in x,y and z, we denote

$$\eta(x,y,z)+\text{cycl.}=\eta(x,y,z)+\eta(y,z,x)+\eta(z,x,y).$$

Proof. Obviously, condition (i) is equivalent to the fact that $[,]_R$ is skew-symmetric. Now we prove that condition (ii) is equivalent to the fact that $[,]_R$ satisfies Jacobi identity. In fact, for all $x,y,z\in\mathfrak{k}$,

$$\begin{aligned}
[[x,y]_R,z]_R &= r([x,y]_R)\cdot z - r(z)\cdot(r(x)\cdot y)+r(z)\cdot(r(y)\cdot x)-\lambda r(z)\cdot[x,y]_{\mathfrak{k}}+\\
&\quad \lambda[r(x)\cdot y,z]_{\mathfrak{k}}-\lambda[r(y)\cdot x,z]_{\mathfrak{k}}+\lambda^2[[x,y]_{\mathfrak{k}},z]_{\mathfrak{k}},\\
[[z,x]_R,y]_R &= r([z,x]_R)\cdot y - r(y)\cdot(r(z)\cdot x)+r(y)\cdot(r(x)\cdot z)-\lambda r(y)\cdot[z,x]_{\mathfrak{k}}+\\
&\quad \lambda[r(z)\cdot x,y]_{\mathfrak{k}}-\lambda[r(x)\cdot z,y]_{\mathfrak{k}}+\lambda^2[[z,x]_{\mathfrak{k}},y]_{\mathfrak{k}},\\
[[y,z]_R,x]_R &= r([y,z]_R)\cdot x - r(x)\cdot(r(y)\cdot z)+r(x)\cdot(r(z)\cdot y)-\lambda r(x)\cdot[y,z]_{\mathfrak{k}}+\\
&\quad \lambda[r(y)\cdot z,x]_{\mathfrak{k}}-\lambda[r(z)\cdot y,x]_{\mathfrak{k}}+\lambda^2[[y,z]_{\mathfrak{k}},x]_{\mathfrak{k}}.
\end{aligned}$$

Then the conclusion follows from the fact that $(\mathfrak{k},\pi)=(\mathfrak{a}^*,\rho^*)$ is a $\mathfrak{g}$-Lie algebra. □

2.2. **Extended $\mathcal{O}$-operators and double Lie brackets.** We will next study the conditions in Proposition 2.9 in order to understand double Lie algebra structures and nonabelian generalized Lax pairs. For this purpose, we introduce the following concepts.

Definition 2.10. Let $(\mathfrak{g},[\ ,\ ]_{\mathfrak{g}})$ be a Lie algebra and let $(\mathfrak{k},[\ ,\ ]_{\mathfrak{k}},\pi)$ be a $\mathfrak{g}$-Lie algebra. Let ν,κ,μ and λ be constants (in $\mathbb{R}$).

(i) A linear map $\beta:\mathfrak{k}\to\mathfrak{g}$ is called **antisymmetric (of mass ν)** if $\nu\beta(x)\cdot y+\nu\beta(y)\cdot x=0$ for any $x,y\in\mathfrak{k}$;

(ii) A linear map $\beta:\mathfrak{k}\to\mathfrak{g}$ is called **$\mathfrak{g}$-invariant (of mass κ)** if $\kappa\beta(\xi\cdot x)=\kappa[\xi,\beta(x)]_{\mathfrak{g}}$, for any $\xi\in\mathfrak{g}, x\in\mathfrak{k}$;

(iii) A linear map $\beta:\mathfrak{k}\to\mathfrak{g}$ is called **equivalent (of mass μ)** if $\mu\beta([x,y]_{\mathfrak{k}})\cdot z=\mu[\beta(x)\cdot y,z]_{\mathfrak{k}}$, for any $x,y,z\in\mathfrak{k}$;

(iv) Let $\beta:\mathfrak{k}\to\mathfrak{g}$ be antisymmetric of mass ν, $\mathfrak{g}$-invariant of mass κ and equivalent of mass μ. Let $r:\mathfrak{k}\to\mathfrak{g}$ be a linear map. The pair (r,β) or simply r is called an **extended $\mathcal{O}$-operator of weight λ with extension β of mass** (ν,κ,μ) if

$$[r(x),r(y)]_{\mathfrak{g}}-r(r(x)\cdot v-r(y)\cdot x+\lambda[x,y]_{\mathfrak{k}})=\kappa[\beta(x),\beta(y)]_{\mathfrak{g}}+\mu\beta([x,y]_{\mathfrak{k}}),\quad \forall x,y\in\mathfrak{k}. \tag{12}$$

(v) A linear map $r:\mathfrak{k}\to\mathfrak{g}$ is called an **$\mathcal{O}$-operator of weight λ** if

$$[r(x),r(y)]_{\mathfrak{g}}=r(r(x)\cdot y-r(y)\cdot x+\lambda[x,y]_{\mathfrak{k}}),\quad \forall x,y\in\mathfrak{k}. \tag{13}$$

(vi) Let $(\mathfrak{k},[\ ,\ ]_{\mathfrak{k}},\pi)$ be the $\mathfrak{g}$-Lie algebra $(\mathfrak{g},[\ ,\ ]_{\mathfrak{g}},\mathrm{ad})$. Then an $\mathcal{O}$-operator $r:\mathfrak{g}\to\mathfrak{g}$ becomes what is known as a **Rota-Baxter operator of weight λ** satisfying

$$[r(x),r(y)]_{\mathfrak{g}}=r([r(x),y]_{\mathfrak{g}}+[x,r(y)]_{\mathfrak{g}}+\lambda[x,y]_{\mathfrak{g}}),\quad \forall x,y\in\mathfrak{g}. \tag{14}$$

A Lie algebra equipped with a Rota-Baxter operator is called a **Rota-Baxter Lie algebra.**

Remark 2.11. (i) We include the parameters ν,κ,μ,λ in the definition in order to uniformly treat the different cases when the parameters vary.

(ii) Rota-Baxter operators on associative algebras were introduced by the mathematician Glenn Baxter [7] in 1960 and have recently found many applications especially in the algebraic approach of Connes and Kreimer to renormalization of quantum field theory [14, 15] . For further details, see the survey articles [20, 23, 42]. See also [6] for the relationship between Rota-Baxter operators on associative algebras and the associative CYBE motivated by the study of this paper.

(iii) If $\lambda \neq 0$, then r is an $\mathcal{O}$-operator of weight λ if and only if r/λ is an $\mathcal{O}$-operator of weight 1.

(iv) When $\lambda = 1$, the difference of the two sides of Eq. (13) has appeared in the work of Y. Kosmann-Schwarzbach and F. Magri under the name Schouten curvature, which is the algebraic version of the contravariant analogue of the Cartan curvature of Lie-algebra valued one-form on a Lie group (see [29] for details).

When $\mathfrak{k}$ in Definition 2.10 is taken to be a vector space regarded as an abelian Lie algebra, then $(\mathfrak{k}, \pi)$ is a $\mathfrak{g}$-Lie algebra means that $\pi : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{k})$ is a linear representation of $\mathfrak{g}$. Thus the above definition has the following variation (with $\nu = \kappa$).

Definition 2.12. Let $\mathfrak{g}$ be a Lie algebra and V be a vector space. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a linear representation of $\mathfrak{g}$. Suppose that $\beta : V \to \mathfrak{g}$ is an antisymmetric of mass κ, $\mathfrak{g}$-invariant of mass κ linear map. Let $r : V \to \mathfrak{g}$ be a linear map. The pair (r, β) or simply r is called an **extended $\mathcal{O}$-operator with extention β of mass κ** if

$$[r(u), r(v)] - r(r(u) \cdot v - r(v) \cdot u) = \kappa[\beta(u), \beta(v)], \quad \forall u, v \in V. \tag{15}$$

When $\kappa = 0$, we obtain the $\mathcal{O}$-operator defined by Kupershmidt [31] and (the operator form of) the classical Yang-Baxter equation (CYBE) of Bordemann [10]. When $\kappa = -1$, Eq. (15) is called the modified classical Yang-Baxter equation (MCYBE) in [10, 28, 43].

The following theorem displays the close connection between extended $\mathcal{O}$-operators and the double Lie algebra structures on $\mathfrak{k}$ in Condition 2.8.

Theorem 2.13. *Let $\mathfrak{g}$ be a Lie algebra and $(\mathfrak{k}, \pi)$ be a $\mathfrak{g}$-Lie algebra. Let $r_\pm : \mathfrak{k} \to \mathfrak{g}$ be two linear maps, $\lambda, \nu, \kappa, \mu \in \mathbb{R}$ and r and β be defined by Eq. (11).*

(i) *Suppose r is an extended $\mathcal{O}$-operator of weight λ with extention β of mass (ν, κ, μ) for $\nu \neq 0$. Then Condition 2.8 holds.*

(ii) *Suppose β satisfies $\beta(\xi \cdot x) = [\xi, \beta(x)]_{\mathfrak{g}}$, for all $\xi \in \mathfrak{g}, x \in \mathfrak{k}$, that is, β is $\mathfrak{g}$-invariant of mass 1 (or equivalently, a $\mathfrak{g}$-module homomorphism). Then r satisfies Eq. (12) for $\kappa = -1$, $\mu = \pm\lambda$ if and only if the following equation holds:*

$$[r_\pm(x), r_\pm(y)]_{\mathfrak{g}} - r_\pm([x, y]_R) = 0, \quad \forall x, y \in \mathfrak{k}. \tag{16}$$

Proof. (i) In order to prove that Eq. (12) implies the Jacobi identity for the bracket $[,]_R$ on $\mathfrak{k}$, it is enough to prove that

$$(k[\beta(x), \beta(y)]_{\mathfrak{g}} + \mu\beta([x, y]_{\mathfrak{k}})) \cdot z + \text{cycl.} = 0.$$

In fact, we will prove that

$$k[\beta(x), \beta(y)]_{\mathfrak{g}} \cdot z + \text{cycl.} = 0 \tag{17}$$

and

$$\mu\beta([x, y]_{\mathfrak{k}}) \cdot z + \text{cycl.} = 0. \tag{18}$$

Eq. (17) has already been proved by Bordemann [10]. In order to be self-contained, we give the details. For any $x, y, z \in \mathfrak{k}$,

$$\begin{aligned} k[\beta(x), \beta(y)]_{\mathfrak{g}} \cdot z &= k\beta(x)(\beta(y) \cdot z) - k\beta(y) \cdot (\beta(x) \cdot z) \\ &= -k\beta(\beta(y) \cdot z) \cdot x - k\beta(\beta(z) \cdot x) \cdot y \quad \text{(by antisymmetry)} \\ &= -k[\beta(y), \beta(z)]_{\mathfrak{g}} \cdot x - k[\beta(z), \beta(x)]_{\mathfrak{g}} \cdot y \quad (\text{by } \mathfrak{g} - \text{invariance}). \end{aligned}$$

So Eq. (17) follows immediately. Moreover,

$$\begin{aligned}
& \mu\beta([x,y]_{\mathfrak{k}})\cdot z = -\mu\beta(z)\cdot[x,y]_{\mathfrak{k}} \quad \text{(by antisymmetry)}\\
= & -\mu[\beta(z)\cdot x, y]_{\mathfrak{k}} - \mu[x,\beta(z)\cdot y]_{\mathfrak{k}}\\
= & \ \mu[\beta(x)\cdot z, y]_{\mathfrak{k}} + \mu[x,\beta(y)\cdot z]_{\mathfrak{k}} \quad \text{(by antisymmetry)}\\
= & \ \mu\beta(x)\cdot[z,y]_{\mathfrak{k}} - \mu[z,\beta(x)\cdot y]_{\mathfrak{k}} + \mu\beta(y)\cdot[x,z]_{\mathfrak{k}} - \mu[\beta(y)\cdot x, z]_{\mathfrak{k}}\\
= & -\mu\beta([z,y]_{\mathfrak{k}})\cdot x - \mu\beta([x,z]_{\mathfrak{k}})\cdot y + 2\mu[\beta(x)\cdot y, z]_{\mathfrak{k}} \quad \text{(by antisymmetry)}\\
= & \ \mu\beta([y,z]_{\mathfrak{k}})\cdot x + \mu\beta([z,x]_{\mathfrak{k}})\cdot y + 2\mu\beta([x,y]_{\mathfrak{k}})\cdot z \quad \text{(by equivalence).}
\end{aligned}$$

Therefore, Eq. (18) holds. So by Proposition 2.9, Condition 2.8 holds.

(ii) A direct computation gives

$$\begin{aligned}
& [(r\pm\beta)(x),(r\pm\beta)(y)]_{\mathfrak{g}} - (r\pm\beta)(r(x)\cdot y - r(y)\cdot x + \lambda[x,y]_{\mathfrak{k}})\\
= & \ [r(x),r(y)]_{\mathfrak{g}} - r(r(x)\cdot y - r(y)\cdot x + \lambda[x,y]_{\mathfrak{k}}) + [\beta(x),\beta(y)]_{\mathfrak{g}} \mp \lambda\beta([x,y]_{\mathfrak{k}}) \pm ([r(x),\beta(y)]_{\mathfrak{g}}\\
& -\beta(r(x)\cdot y) + [\beta(x),r(y)]_{\mathfrak{g}} + \beta(r(y)\cdot x))\\
= & \ [r(x),r(y)]_{\mathfrak{g}} - r(r(x)\cdot y - r(y)\cdot x + \lambda[x,y]_{\mathfrak{k}}) + [\beta(x),\beta(y)]_{\mathfrak{g}} \mp \lambda\beta([x,y]_{\mathfrak{k}}),
\end{aligned}$$

where the last equality follows from $\mathfrak{g}$-invariance of mass 1. So (ii) holds. □

Remark 2.14. When the bracket $[,]_{\mathfrak{k}}$ on $\mathfrak{k}$ is trivial and $\kappa = -1$, Proposition 2.9 and Theorem 2.13 give Theorem 2.18 in [10].

The following results give the relations of $\mathcal{O}$-operators with Eq. (16) and extended $\mathcal{O}$-operators.

Theorem 2.15. *Let $\mathfrak{g}$ be a Lie algebra and $(\mathfrak{k},\pi)$ be a $\mathfrak{g}$-Lie algebra. Let $r_\pm : \mathfrak{k}\to\mathfrak{g}$ be two linear maps and let $\lambda\in\mathbb{R}$ and r and β be defined by Eq. (11). Suppose that β is antisymmetric of mass $\nu\neq 0$, $\mathfrak{g}$-invariant of mass $\kappa\neq 0$ and equivalent of mass λ.*

(i) *$(\mathfrak{k}_\pm,[\ ,\ ]_\pm,\pi)$ are $\mathfrak{g}$-Lie algebras, where $(\mathfrak{k}_\pm,[\ ,\ ]_\pm)$ are the new Lie algebra structures on $\mathfrak{k}$ defined by*

$$[x,y]_\pm \equiv \lambda[x,y]_{\mathfrak{k}} \pm 2\beta(x)\cdot y, \quad \forall x,y\in\mathfrak{k}. \tag{19}$$

(ii) *r is an extended $\mathcal{O}$-operator of weight λ with extention β of mass $(\nu,-1,\pm\lambda)$ for $\nu\neq 0$ if and only if $r_\pm : \mathfrak{k}_\mp\to\mathfrak{g}$ is an $\mathcal{O}$-operators of weight 1, where $\mathfrak{k}_\mp$ is equipped with the Lie bracket $[,]_\mp$ defined by Eq. (19).*

Proof. (i) Since β is antisymmetric, $[,]_\pm$ is antisymmetric. Moreover, for any $x,y,z\in\mathfrak{k}$, we have

$$\begin{aligned}
& [[x,y]_\pm,z]_\pm + \text{cycl.} = [\lambda[x,y]_{\mathfrak{k}} \pm 2\beta(x)\cdot y, z]_\pm + \text{cycl.}\\
= & \ (\lambda^2[[x,y]_{\mathfrak{k}},z]_{\mathfrak{k}} \pm 2\lambda[\beta(x)\cdot y,z]_{\mathfrak{k}} \pm 2\lambda\beta([x,y]_{\mathfrak{k}})\cdot z + 4\beta(\beta(x)\cdot y)\cdot z) + \text{cycl.}\\
= & \ (\lambda^2[[x,y]_{\mathfrak{k}},z]_{\mathfrak{k}} \pm 4\lambda\beta([x,y]_{\mathfrak{k}})\cdot z + 4[\beta(x),\beta(y)]_{\mathfrak{g}}\cdot z) + \text{cycl.},
\end{aligned}$$

where the last equality follows from the $\mathfrak{g}$-invariance of mass $\kappa\neq 0$ and equivalence of mass λ. So by Theorem 2.13 the Jacobi identity for the bracket $[,]_\pm$ on $\mathfrak{k}$ holds. Moreover, for any $\xi\in\mathfrak{g}$, we have

$$\begin{aligned}
& \xi\cdot[x,y]_\pm = \lambda\xi\cdot[x,y]_{\mathfrak{k}} \pm 2\xi\cdot(\beta(x)\cdot y)\\
= & \ \lambda[\xi\cdot x,y]_{\mathfrak{k}} + \lambda[x,\xi\cdot y]_{\mathfrak{k}} \pm 2\beta(\xi\cdot x)\cdot y \pm 2\beta(x)\cdot(\xi\cdot y) \quad (\text{by } \mathfrak{g}-\text{invariance})\\
= & \ [\xi\cdot x,y]_\pm + [x,\xi\cdot y]_\pm.
\end{aligned}$$

So $(\mathfrak{k}_\pm,\pi)$ equipped with the bracket $[,]_\pm$ on $\mathfrak{k}$ is a $\mathfrak{g}$-Lie algebra.

(ii) The last conclusion follows from Theorem 2.13, Item (i) and the following computations:

$$\begin{aligned}
& [r_\pm(x), r_\pm(y)]_{\mathfrak{g}} - r_\pm([x,y]_R) \\
= \ & [r_\pm(x), r_\pm(y)]_{\mathfrak{g}} - r_\pm(r_\pm(x)\cdot y - r_\pm(y)\cdot x + \lambda[x,y]_{\mathfrak{k}} \mp \beta(x)\cdot y \pm \beta(y)\cdot x) \\
= \ & [r_\pm(x), r_\pm(y)]_{\mathfrak{g}} - r_\pm(r_\pm(x)\cdot y - r_\pm(y)\cdot x + [x,y]_\mp) \quad \text{(by antisymmetry).}
\end{aligned}$$

□

When $\mathfrak{k}$ in Theorem 2.15 is taken to be a vector space regarded as an abelian Lie algebra, we obtain the following conclusions.

Corollary 2.16. *Let $\mathfrak{g}$ be a Lie algebra and V be a vector space. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a linear representation of $\mathfrak{g}$. Suppose that $\beta : V \to \mathfrak{g}$ is antisymmetric of mass $\kappa \neq 0$ and $\mathfrak{g}$-invariant of mass $\kappa \neq 0$.*

(i) *$(V_\pm, [\ ,\ ]_\pm, \rho)$ are $\mathfrak{g}$-Lie algebras, where $(V_\pm, [\ ,\ ]_\pm)$ are the Lie algebra structures on V defined by*

$$[u,v]_\pm \equiv \pm 2\beta(u)\cdot v, \quad \forall u,v \in V. \tag{20}$$

(ii) *Let $r : V \to \mathfrak{g}$ be a linear map. Then r is an extended $\mathcal{O}$-operator with extention β of mass -1 if and only if $r \pm \beta : V_\mp \to \mathfrak{g}$ are $\mathcal{O}$-operators of weight 1, where $V_\mp$ are equipped with the Lie brackets $[\ ,\ ]_\mp$ defined by Eq. (20).*

2.3. **Adjoint representations and Baxter Lie algebras.** We now consider the case of adjoint representations. If $\mathfrak{k} = \mathfrak{g}$ with the trivial Lie bracket and $\pi = \mathrm{ad}$, then by Proposition 2.9, Theorem 2.13 and Theorem 2.15 we have the following conclusion.

Proposition 2.17. *Let $\mathfrak{g}$ be a Lie algebra and $R, \beta : \mathfrak{g} \to \mathfrak{g}$ be two linear maps. Let β be antisymmetric of mass κ and $\mathfrak{g}$-invariant of mass κ, i.e., the following equation holds:*

$$\kappa\beta([x,y]) = \kappa[\beta(x), y] = \kappa[x, \beta(y)], \quad \forall x,y \in \mathfrak{g}. \tag{21}$$

Suppose that R is an extended $\mathcal{O}$-operator with extention β of mass κ, i.e., the following equation holds:

$$[R(x), R(y)] - R([R(x), y] + [x, R(y)]) = \kappa[\beta(x), \beta(y)], \quad \forall x,y \in \mathfrak{g}. \tag{22}$$

Then the product

$$[x,y]_R = [R(x), y] + [x, R(y)], \quad \forall x,y \in \mathfrak{g},$$

defines a Lie bracket on $\mathfrak{g}$. On the other hand, if β satisfies Eq. (21) for $\kappa \neq 0$, then $(\mathfrak{g}_\pm, [\ ,\ ]_\pm, \mathrm{ad})$ are $\mathfrak{g}$-Lie algebras, where $(\mathfrak{g}_\pm, [\ ,\ ]_\pm)$ are the new Lie algebra structures defined by

$$[x,y]_\pm \equiv \pm 2[\beta(x), y], \quad \forall x,y \in \mathfrak{g}. \tag{23}$$

Moreover, R is an extended $\mathcal{O}$-operator with extention β of mass -1, i.e., Eq. (22) holds for $\kappa = -1$, if and only if $R \pm \beta : \mathfrak{g}_\mp \to \mathfrak{g}$ are $\mathcal{O}$-operators of weight 1, where $\mathfrak{g}_\mp$ are equipped with the Lie brackets $[\ ,\ ]_\mp$ defined by Eq. (23).

Remark 2.18. Let $\mathfrak{g}$ be a Lie algebra. A linear endomorphism β of $\mathfrak{g}$ satisfying Eq. (21) for $\kappa \neq 0$ is called an **intertwining operator** in [40], where it is used to construct **compatible Poisson brackets**. If $\beta : \mathfrak{g} \to \mathfrak{g}$ is an intertwining operator on $\mathfrak{g}$, then it is also an **averaging operator** [2, 42] in the Lie algebraic context, namely,

$$[\beta(x), \beta(y)] = \beta([x, \beta(y)]) = \beta([\beta(x), y]), \quad \forall x,y \in \mathfrak{g},$$

and is a **Nijenhuis tensor**, namely,

$$[\beta(x), \beta(y)] + \beta^2([x,y]) = \beta([\beta(x), y] + [x, \beta(y)]), \quad \forall x,y \in \mathfrak{g}. \tag{24}$$

Let the $\mathfrak{g}$-Lie algebra $(\mathfrak{k},\pi)$ be $(\mathfrak{g},\mathrm{ad})$. It is obvious that $\beta=\mathrm{id}:\mathfrak{g}\to\mathfrak{g}$ satisfies the conditions of Proposition 2.9, Theorem 2.13 and Theorem 2.15 and in this case, Eq. (12) takes the following form (set $r=R$):

$$[R(x),R(y)]-R([R(x),y]+[x,R(y)]+\hat{\lambda}[x,y])=\hat{\kappa}[x,y],\quad \forall x,y\in\mathfrak{g}, \tag{25}$$

for $\hat{\lambda}=\lambda$ and $\hat{\kappa}=\kappa+\mu$. When $\hat{\kappa}=-1\pm\hat{\lambda}$, by Theorem 2.15, R satisfies Eq. (25) if and only if $R\pm\mathrm{id}$ is a Rota-Baxter operator of weight $\hat{\lambda}\mp 2$. Note that when $\hat{\lambda}=0$, Eq. (25) takes the following form

$$[R(x),R(y)]-R([R(x),y]+[x,R(y)])=\kappa[x,y],\quad \forall x,y\in\mathfrak{g}, \tag{26}$$

for $\kappa=\hat{\kappa}$. When $\kappa=-1$, Eq. (26) becomes

$$[R(x),R(y)]-R([R(x),y]+[x,R(y)])=-[x,y],\quad \forall x,y\in\mathfrak{g}. \tag{27}$$

A Lie algebra equipped with a linear endomorphism satisfying Eq. (27) is called a **Baxter Lie algebra** in [10]. We note the difference between a Baxter Lie algebra and a Rota-Baxter Lie algebra defined in Definition 2.10. Moreover, the equivalence of the facts that R satisfies Eq. (27) and $R\pm\mathrm{id}$ is a Rota-Baxter operator of weight ∓ 2 was pointed out in [19, 43].

3. Extended $\mathcal{O}$-operators, the extended CYBE and type II quasitriangular Lie bialgebras

In this section, we define the extended CYBE and apply the study in Section 2 to investigate the relationship between extended $\mathcal{O}$-operators and the extended CYBE. We also introduce the concept of type II quasitriangular Lie bialgebras from type II CYBE as a parallel concept of quasitriangular Lie bialgebras from CYBE. We then explicitly describe the Drinfeld's doubles and Manin triples of type II quasitriangular Lie bialgebras.

3.1. Lie bialgebras and the extended CYBE.

We recall the following concepts [13].

Definition 3.1. Let $\mathfrak{g}$ be a Lie algebra.

(i) A **Lie bialgebra** structure on $\mathfrak{g}$ is a skew-symmetric $\mathbb{R}$-linear map $\delta:\mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}$, called **co-commutator**, such that $(\mathfrak{g},\delta)$ is a Lie coalgebra and δ is a 1-cocycle of $\mathfrak{g}$ with coefficients in $\mathfrak{g}\otimes\mathfrak{g}$, that is, it satisfies the following equation:

$$\delta([x,y])=(\mathrm{ad}(x)\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{ad}(x))\delta(y)-(\mathrm{ad}(y)\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{ad}(y))\delta(x),\quad \forall x,y\in\mathfrak{g}.$$

(ii) A Lie bialgebra $(\mathfrak{g},\delta)$ is called **coboundary** if δ is a 1-coboundary, that is, there exists an $r\in\mathfrak{g}\otimes\mathfrak{g}$ such that

$$\delta(x)=(\mathrm{ad}(x)\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{ad}(x))r,\quad \forall x\in\mathfrak{g}. \tag{28}$$

We usually denote the coboundary Lie bialgebra by $(\mathfrak{g},r)$ or simply $\mathfrak{g}$.

(iii) A **Manin triple** is a triple $(\mathfrak{a},\mathfrak{a}_+,\mathfrak{a}_-)$ of Lie algebras together with a nondegenerate symmetric invariant bilinear form $\mathfrak{B}(\ ,\ )$ on $\mathfrak{a}$, such that
  (a) $\mathfrak{a}_+$ and $\mathfrak{a}_-$ are Lie subalgebras of $\mathfrak{a}$;
  (b) $\mathfrak{a}=\mathfrak{a}_+\oplus\mathfrak{a}_-$ as vector spaces;
  (c) $\mathfrak{a}_+$ and $\mathfrak{a}_-$ are isotropic for $\mathfrak{B}(\ ,\ )$.

We recall the following basic results on Lie bialgebras and Manin triples.

Proposition 3.2. ([17]) *Let $(\mathfrak{g},\delta)$ be a Lie bialgebra. Let $\mathcal{D}(\mathfrak{g})\equiv\mathfrak{g}\oplus\mathfrak{g}^*$. Then $(\mathcal{D}(\mathfrak{g}),\mathfrak{g},\mathfrak{g}^*)$ is a Manin triple with respect to the bilinear form*

$$\mathfrak{B}_p((x,a^*),(y,b^*))=\langle a^*,y\rangle+\langle x,b^*\rangle,\quad \forall x,y\in\mathfrak{g},\ a^*,b^*\in\mathfrak{g}^*, \tag{29}$$

on $\mathcal{D}(\mathfrak{g})$. Explicitly, the Lie algebra structure on $\mathcal{D}(\mathfrak{g})$ is given by

$$[(x,a^*),(y,b^*)]_{\mathcal{D}(\mathfrak{g})} = ([x,y]+\mathrm{ad}^*(a^*)y-\mathrm{ad}^*(b^*)x, [a^*,b^*]_\delta+\mathrm{ad}^*(x)b^*-\mathrm{ad}^*(y)a^*),\ \forall x,y\in\mathfrak{g}, a^*,b^*\in\mathfrak{g}^*, \tag{30}$$

where the Lie algebra structure $[\ ,\ ]_\delta$ on $\mathfrak{g}^*$ is defined by

$$\langle [a^*,b^*]_\delta, x\rangle = \langle a^*\otimes b^*, \delta(x)\rangle, \quad \forall x\in\mathfrak{g}, a^*,b^*\in\mathfrak{g}^*. \tag{31}$$

$\mathcal{D}(\mathfrak{g})$ is called the **Drinfeld's double** for the Lie bialgebra $(\mathfrak{g}, r)$.

Proposition 3.3. ([13]) *Let $\mathfrak{g}$ be a Lie algebra and $r\in\mathfrak{g}\otimes\mathfrak{g}$. The linear map δ defined by Eq. (28) is the commutator of a Lie bialgebra structure on $\mathfrak{g}$ if and only if the following conditions are satisfied for all $x\in\mathfrak{g}$:*

(i) $(\mathrm{ad}(x)\otimes \mathrm{id}+\mathrm{id}\otimes\mathrm{ad}(x))(r+\sigma(r))=0$, *that is, the symmetric part of r is invariant.*
(ii) $(\mathrm{ad}(x)\otimes\mathrm{id}\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{ad}(x)\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{ad}(x))([r_{12},r_{13}]+[r_{12},r_{23}]+[r_{13},r_{23}])=0.$

Here $\sigma:\mathfrak{g}^{\otimes 2}\to\mathfrak{g}^{\otimes 2}$ is the twisting operator defined by

$$\sigma(x\otimes y) = y\otimes x, \quad \forall x,y\in\mathfrak{g}.$$

In the following we call $r=\sum\limits_i a_i\otimes b_i\in\mathfrak{g}^{\otimes 2}$ **skew-symmetric** (resp. **symmetric**) if $r=-\sigma(r)$ (resp. $r=\sigma(r)$). Moreover, we use the notations (in the universal enveloping algebra $U(\mathfrak{g})$):

$$r_{12}=\sum_i a_i\otimes b_i\otimes 1,\quad r_{13}=\sum_i a_i\otimes 1\otimes b_i,\quad r_{23}=\sum_i 1\otimes a_i\otimes b_i,$$

and

$$[r_{12},r_{13}]=\sum_{i,j}[a_i,a_j]\otimes b_i\otimes b_j,\ [r_{13},r_{23}]=\sum_{i,j}a_i\otimes a_j\otimes[b_i,b_j],\ [r_{23},r_{12}]=\sum_{i,j}a_j\otimes[a_i,b_j]\otimes b_i.$$

The equation

$$\mathbf{C}(r)\equiv[r_{12},r_{13}]+[r_{12},r_{23}]+[r_{13},r_{23}]=0 \tag{32}$$

is called the (tensor form of) the **classical Yang-Baxter equation** (CYBE). One should not confuse it with the (operator form of) CYBE of Bordemann [10], though under certain conditions the former is equivalent to a particular case of the later that we will elaborate next.

A coboundary Lie bialgebra $(\mathfrak{g}, r)$ arising from a solution of CYBE is said to be **quasitriangular**, whereas a coboundary Lie bialgebra $(\mathfrak{g}, r)$ arising from a skew-symmetric solution of CYBE is said to be **triangular** [9, 13]. Note that for any coboundary Lie bialgebra $(\mathfrak{g}, r)$, the condition (i) in Proposition 3.3 holds automatically.

For any $r=\sum\limits_i a_i\otimes b_i\in\mathfrak{g}\otimes\mathfrak{g}$, we set

$$r_{21}=\sum_i b_i\otimes a_i\otimes 1,\quad r_{32}=\sum_i 1\otimes b_i\otimes a_i,\quad r_{31}=\sum_i b_i\otimes 1\otimes a_i.$$

Moreover, we set

$$[(a_1\otimes a_2\otimes a_3),(b_1\otimes b_2\otimes b_3)]=[a_1,b_1]\otimes[a_2,b_2]\otimes[a_3,b_3],\quad \forall\, a_i,b_i\in\mathfrak{g}, i=1,2,3.$$

Definition 3.4. Let $\mathfrak{g}$ be a Lie algebra. Fix $\epsilon\in\mathbb{R}$. The equation

$$[r_{12},r_{13}]+[r_{12},r_{23}]+[r_{13},r_{23}]=\epsilon[(r_{13}+r_{31}),(r_{23}+r_{32})] \tag{33}$$

is called the **extended classical Yang-Baxter equation of mass ϵ** (or **ECYBE of mass ϵ** in short).

Remark 3.5. (i) When $\epsilon = 0$ or r is skew-symmetric, then the ECYBE of mass ϵ is the same as the CYBE in Eq. (32):

(ii) If the symmetric part β of r is invariant, by the proof of Theorem 3.9 below, for any $a^*, b^*, c^* \in \mathfrak{g}^*$, we have

$$\begin{aligned}
\langle [r_{13}+r_{31}, r_{23}+r_{32}], a^*\otimes b^*\otimes c^*\rangle &= \langle 4[\beta(a^*),\beta(b^*)], c^*\rangle = \langle 4\beta(\mathrm{ad}^*(\beta(a^*))b^*), c^*\rangle\\
&= \langle [r_{23}+r_{32}, r_{12}+r_{21}], a^*\otimes b^*\otimes c^*\rangle\\
\langle [r_{13}+r_{31}, r_{23}+r_{32}], a^*\otimes b^*\otimes c^*\rangle &= \langle 4[\beta(a^*),\beta(b^*)], c^*\rangle = \langle -4\beta(\mathrm{ad}^*(\beta(b^*))a^*), c^*\rangle\\
&= \langle [r_{12}+r_{21}, r_{13}+r_{31}], a^*\otimes b^*\otimes c^*\rangle.
\end{aligned}$$

So in this case, the ECYBE of mass ϵ is equivalent to either one of the following two equations:

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = \epsilon[r_{23}+r_{32}, r_{12}+r_{21}],$$

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = \epsilon[r_{12}+r_{21}, r_{13}+r_{31}].$$

3.2. **Extended $\mathcal{O}$-operators and the ECYBE.** We now study the relationship between extended $\mathcal{O}$-operators and solutions of the ECYBE, generalizing the well-known relationship between the operator form and tensor form of the CYBE [29].

Let $\mathfrak{g}$ be a Lie algebra and $r \in \mathfrak{g}\otimes\mathfrak{g}$. Since $\mathfrak{g}$ is assumed to be finite-dimensional, we will be able to identify r with the linear map $r : \mathfrak{g}^* \to \mathfrak{g}$ through

$$\langle r(a^*), b^*\rangle = \langle a^*\otimes b^*, r\rangle, \quad \forall a^*, b^* \in \mathfrak{g}^*. \tag{34}$$

We will do this throughout the rest of the paper. Moreover, $r^t : \mathfrak{g}^* \to \mathfrak{g}$ is defined as

$$\langle a^*, r^t(b^*)\rangle = \langle a^*\otimes b^*, r\rangle, \quad \forall a^*, b^* \in \mathfrak{g}^*.$$

Note that r^t is just the linear map (from $\mathfrak{g}^*$ to $\mathfrak{g}$) induced by $\sigma(r)$. We also use the following notations:

$$\alpha = (r-\sigma(r))/2 = (r-r^t)/2, \quad \beta = (r+\sigma(r))/2 = (r+r^t)/2, \tag{35}$$

that is, α and β are the **skew-symmetric part** and **symmetric part** of r respectively, and in this case $r = \alpha+\beta$ and $r^t = -\alpha+\beta$.

Lemma 3.6. *Let $\mathfrak{g}$ be a Lie algebra and $\beta \in \mathfrak{g}\otimes\mathfrak{g}$ be symmetric. Then the following conditions are equivalent.*

(i) *$\beta \in \mathfrak{g}\otimes\mathfrak{g}$ is invariant, that is, $(\mathrm{ad}(x)\otimes \mathrm{id} + \mathrm{id}\otimes\mathrm{ad}(x))\beta = 0$, for any $x \in \mathfrak{g}$;*

(ii) *$\beta : \mathfrak{g}^* \to \mathfrak{g}$ is antisymmetry, that is, $\mathrm{ad}^*(\beta(a^*))b^* + \mathrm{ad}^*(\beta(b^*))a^* = 0$, for any $a^*, b^* \in \mathfrak{g}^*$;*

(iii) *$\beta : \mathfrak{g}^* \to \mathfrak{g}$ is $\mathfrak{g}$-invariant, that is, $\beta(\mathrm{ad}^*(x)a^*) = [x, \beta(a^*)]$, for any $x \in \mathfrak{g}$, $a^* \in \mathfrak{g}^*$.*

Proof. Bordemann in [10] pointed out the equivalence of (ii) and (iii). For completeness, we shall prove (i)⇔(ii) and (i)⇔(iii). In fact, for any $x \in \mathfrak{g}$, $a^*, b^* \in \mathfrak{g}^*$,

$$\begin{aligned}
\langle (\mathrm{ad}(x)\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{ad}(x))\beta, a^*\otimes b^*\rangle &= \langle \beta, -(\mathrm{ad}^*(x)a^*)\otimes b^*\rangle + \langle \beta, -a^*\otimes(\mathrm{ad}^*(x)b^*)\rangle\\
&= \langle a^*, [x,\beta(b^*)]\rangle + \langle [x,\beta(a^*)], b^*\rangle \quad \text{(by symmetry)}\\
&= \langle \mathrm{ad}^*(\beta(b^*))a^* + \mathrm{ad}^*(\beta(a^*))b^*, x\rangle.
\end{aligned}$$

So (i)⇔(ii). Moreover,

$$\begin{aligned}
\langle (\mathrm{ad}(x)\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{ad}(x))\beta, a^*\otimes b^*\rangle &= \langle \beta, -(\mathrm{ad}^*(x)a^*)\otimes b^*\rangle + \langle \beta, -a^*\otimes(\mathrm{ad}^*(x)b^*)\rangle\\
&= \langle -\beta(\mathrm{ad}^*(x)a^*) + [x,\beta(a^*)], b^*\rangle.
\end{aligned}$$

So (i)⇔(iii). □

Note that the condition (i) in Lemma 3.6 is exactly the condition (i) of Proposition 3.3.

Lemma 3.7. ([29]) *Let $\mathfrak{g}$ be a Lie algebra and $r \in \mathfrak{g}\otimes\mathfrak{g}$. Let $\alpha, \beta : \mathfrak{g}^* \to \mathfrak{g}$ be the two linear maps given by Eq. (35). Then the bracket $[,]_\delta$ defined by Eq. (31) satisfies*

$$[a^*, b^*]_\delta = \mathrm{ad}^*(r(a^*))b^* + \mathrm{ad}^*(r^t(b^*))a^*, \quad \forall a^*, b^* \in \mathfrak{g}^*. \tag{36}$$

Moreover, if the symmetric part β of r is invariant, then

$$[a^*, b^*]_\delta = \mathrm{ad}^*(\alpha(a^*))b^* - \mathrm{ad}^*(\alpha(b^*))a^*, \quad \forall a^*, b^* \in \mathfrak{g}^*. \tag{37}$$

We supply a proof to be self-contained.

Proof. Let $\{e_i\}_{1\le i\le \dim\mathfrak{g}}$ be a basis of $\mathfrak{g}$ and $\{e_i^*\}_{1\le i\le\dim\mathfrak{g}}$ be its dual basis. Then the first conclusion holds due to the following equations:

$$\begin{aligned}
[e_k^*, e_l^*]_\delta &= \sum_s \langle e_k^*\otimes e_l^*, \delta(e_s)\rangle e_s^* = \sum_s \langle e_k^*\otimes e_l^*, (\mathrm{ad}(e_s)\otimes \mathrm{id} + \mathrm{id}\otimes \mathrm{ad}(e_s))r\rangle e_s^* \\
&= \sum_{s,t} (a_{tl}c_{st}^k + a_{kt}c_{st}^l)e_s^* = \mathrm{ad}^*(r(e_k^*))e_l^* + \mathrm{ad}^*(r^t(e_l^*))e_k^*.
\end{aligned}$$

The last conclusion follows from Lemma 3.6. □

The above lemma motivates us to apply the study in Section 2. More precisely, we have the following results.

Proposition 3.8. *Let $\mathfrak{g}$ be a Lie algebra and $r \in \mathfrak{g}\otimes\mathfrak{g}$. Let $\alpha, \beta : \mathfrak{g}^* \to \mathfrak{g}$ be two linear maps given by Eq. (35). Suppose that β, regarded as an element of $\mathfrak{g}\otimes\mathfrak{g}$, is invariant.*

(i) *$(\mathfrak{g}, r)$ becomes a (coboundary) Lie bialgebra if α is an extended $\mathcal{O}$-operator with extention β of mass $\kappa \in \mathbb{R}$, namely the following equation holds:*

$$[\alpha(a^*), \alpha(b^*)] - \alpha(\mathrm{ad}^*(\alpha(a^*))b^* - \mathrm{ad}^*(\alpha(b^*))a^*) = \kappa[\beta(a^*), \beta(b^*)], \quad \forall a^*, b^* \in \mathfrak{g}^*. \tag{38}$$

(ii) ([29]) *The following conditions are equivalent:*

(a) *α is an extended $\mathcal{O}$-operator with extention β of mass -1, i.e., Eq. (38) (with $\kappa = -1$) holds;*

(b) *r (resp. $-r^t$) satisfies the following equation:*

$$[r(a^*), r(b^*)] = r([a^*, b^*]_\delta), \quad \forall a^*, b^* \in \mathfrak{g}^* \tag{39}$$

$$(\text{resp.} \quad [(-r^t)(a^*), (-r^t)(b^*)] = (-r^t)([a^*, b^*]_\delta), \quad \forall a^*, b^* \in \mathfrak{g}^*); \tag{40}$$

(c) *r (resp. $-r^t$) is an $\mathcal{O}$-operator of weight 1, that is, r (resp. $-r^t$) satisfies the following equation:*

$$[r(a^*), r(b^*)] = r(\mathrm{ad}^*(r(a^*))b^* - \mathrm{ad}^*(r(b^*))a^* + [a^*, b^*]_-), \quad \forall a^*, b^* \in \mathfrak{g}^*, \tag{41}$$

$$(\text{resp.} \ [(-r^t)(a^*), (-r^t)(b^*)] = (-r^t)(\mathrm{ad}^*((-r^t)(a^*))b^* - \mathrm{ad}^*((-r^t)(b^*))a^* + [a^*, b^*]_+), \forall a^*, b^* \in \mathfrak{g}^*) \tag{42}$$

where the brackets $[,]_\pm$ on $\mathfrak{g}^*$ are defined by

$$[a^*, b^*]_\pm \equiv \pm 2\mathrm{ad}^*(\beta(a^*))b^*, \quad \forall a^*, b^* \in \mathfrak{g}^*, \tag{43}$$

and $(\mathfrak{g}^*, \mathrm{ad}^*)$ equipped with the brackets $[,]_\pm$ on $\mathfrak{g}^*$ are $\mathfrak{g}$-Lie algebras.

Proof. (i) By Lemma 3.7, we see that $(\mathfrak{g}, r)$ becomes a (coboundary) Lie bialgebra if the bracket $[\,,\,]_\delta$ defined by Eq. (36) is a Lie structure on $\mathfrak{g}^*$. Further by Lemma 3.6, β is antisymmetric of mass $\nu \neq 0$ and $\mathfrak{g}$-invariant of mass $\kappa \neq 0$. Then the conclusion follows from Theorem 2.13.(i) by setting $(\mathfrak{k}, \pi) = (\mathfrak{g}^*, \mathrm{ad}^*)$ with trivial Lie bracket, $r_+ = r$ and $r_- = -r^t$.

(ii) It follows from Theorem 2.13 and Theorem 2.15 by setting $(\mathfrak{k}, \pi) = (\mathfrak{g}^*, \mathrm{ad}^*)$ with trivial Lie bracket, $r_+ = r$ and $r_- = -r^t$. □

The following theorem establishes a close relationship between extended $\mathcal{O}$-operators on a Lie algebra $\mathfrak{g}$ and solutions of the ECYBE in $\mathfrak{g}$.

Theorem 3.9. *Let $\mathfrak{g}$ be a Lie algebra and let $r \in \mathfrak{g} \otimes \mathfrak{g}$ which is identified as a linear map from $\mathfrak{g}^*$ to $\mathfrak{g}$. Define α and β by Eq. (35). Suppose that the symmetric part β of r is invariant. Then r is a solution of ECYBE of mass $\frac{\kappa+1}{4}$:*

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = \frac{\kappa + 1}{4}[(r_{13} + r_{31}), (r_{23} + r_{32})]$$

if and only if α is an extended $\mathcal{O}$-operator with extention β of mass κ, i.e., Eq. (38) holds.

Proof. Let $r = \sum\limits_{i,j} u_i \otimes v_i \in \mathfrak{g} \otimes \mathfrak{g}$ for $u_i, v_i \in \mathfrak{g}$, then

$$\begin{aligned}
\langle [r_{12}, r_{13}], a^* \otimes b^* \otimes c^* \rangle &= \sum_{i,j} \langle [u_i, u_j], a^* \rangle \langle v_i, b^* \rangle \langle v_j, c^* \rangle = \langle -r(\mathrm{ad}^*(r^t(b^*))a^*), c^* \rangle, \\
\langle [r_{12}, r_{23}], a^* \otimes b^* \otimes c^* \rangle &= \sum_{i,j} \langle u_i, a^* \rangle \langle [v_i, u_j], b^* \rangle \langle v_j, c^* \rangle = \langle -r(\mathrm{ad}^*(r(a^*))b^*), c^* \rangle, \\
\langle [r_{13}, r_{23}], a^* \otimes b^* \otimes c^* \rangle &= \sum_{i,j} \langle u_i, a^* \rangle \langle u_j, b^* \rangle \langle [v_i, v_j], c^* \rangle = \langle [r(a^*), r(b^*)], c^* \rangle.
\end{aligned}$$

Therefore, r is a solution of CYBE in $\mathfrak{g}$ if and only if Eq. (39) holds, i.e.,

$$[r(a^*), r(b^*)] = r(\mathrm{ad}^*(r(a^*))b^* + \mathrm{ad}^*(r^t(b^*))a^*), \quad \forall a^*, b^* \in \mathfrak{g}^*.$$

Therefore, by Proposition 3.8, for any $a^*, b^*, c^* \in \mathfrak{g}^*$, we have that

$$\begin{aligned}
&\langle [\alpha(a^*), \alpha(b^*)] - \alpha(\mathrm{ad}^*(\alpha(a^*))b^* - \mathrm{ad}^*(\alpha(b^*))a^*) - \kappa[\beta(a^*), \beta(b^*)], c^* \rangle \\
= \;& \langle [\alpha(a^*), \alpha(b^*)] - \alpha(\mathrm{ad}^*(\alpha(a^*))b^* - \mathrm{ad}^*(\alpha(b^*))a^*) + [\beta(a^*), \beta(b^*)] - (\kappa + 1)[\beta(a^*), \beta(b^*)], c^* \rangle \\
= \;& \langle [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}], a^* \otimes b^* \otimes c^* \rangle - (\kappa + 1)\langle [\beta_{13}, \beta_{23}], a^* \otimes b^* \otimes c^* \rangle \\
= \;& \langle [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] - (\kappa + 1)[\frac{r_{13} + r_{31}}{2}, \frac{r_{23} + r_{32}}{2}], a^* \otimes b^* \otimes c^* \rangle.
\end{aligned}$$

So r is a solution of the ECYBE of mass $(\kappa + 1)/4$ if and only if α is an extended $\mathcal{O}$-operator with extention β of mass κ. □

Therefore by Proposition 3.8 and Theorem 3.9 (for $\kappa = -1$), we have the following conclusion:

Corollary 3.10. ([29]) *Let $\mathfrak{g}$ be a Lie algebra and $r \in \mathfrak{g} \otimes \mathfrak{g}$. Let $\alpha, \beta : \mathfrak{g}^* \to \mathfrak{g}$ be two linear maps given by Eq. (35). Suppose that β, regarded as an element of $\mathfrak{g} \otimes \mathfrak{g}$, is invariant. Then the following conditions are equivalent:*

(i) *r is a solution of the CYBE;*
(ii) *$(\mathfrak{g}, r)$ is a quasitriangular Lie bialgebra;*
(iii) *r (resp. $-r^t$) is an $\mathcal{O}$-operator of weight 1, that is, r (resp. $-r^t$) satisfies Eq. (41)) (resp. Eq. (42)) with $\mathfrak{g}^*$ equipped with the bracket $[\,,\,]_-$ (resp. $[\,,\,]_+$) defined by Eq. (43).*
(iv) *α is an extended $\mathcal{O}$-operator with extention β of mass -1, i.e., α and β satisfy Eq. (38) with $k = -1$;*
(v) *r (resp. $-r^t$) satisfies Eq. (39) (resp. Eq. (40)).*

3.3. **Extended $\mathcal{O}$-operators (of mass 1) and type II CYBE.** Proposition 3.8 and Theorem 3.9 reveal close connections of extended $\mathcal{O}$-operators $\alpha : \mathfrak{g}^* \to \mathfrak{g}$ (defined by Eq. (38)) with coboundary Lie bialgebras and ECYBE. Thus we would like to study these operators in more detail. Note that, for $\kappa = \eta^2\kappa'$ with $\kappa, \kappa' \in \mathbb{R}$ and $\eta \in \mathbb{R}^\times$, α is an extended $\mathcal{O}$-operator with extention β of mass κ if and only if α is an extended $\mathcal{O}$-operator with extention $\eta\beta$ of mass κ'. Thus we only need to consider the cases when $\kappa = 0, 1, -1$.

The case of $\kappa = -1$ is considered in Corollary 3.10. The case of $\kappa = 0$ has been considered by Kupershmidt [31] as remarked before. So we will next focus on the case when $\kappa = 1$:

$$[\alpha(a^*), \alpha(b^*)] - \alpha(\mathrm{ad}^*(\alpha(a^*))b^* - \mathrm{ad}^*(\alpha(b^*))a^*) = [\beta(a^*), \beta(b^*)], \quad \forall a^*, b^* \in \mathfrak{g}^*. \tag{44}$$

Note here β regarded as an element of $\mathfrak{g} \otimes \mathfrak{g}$ is invariant (Lemma 3.6).

Definition 3.11. Let $\mathfrak{g}$ be a Lie algebra and $r \in \mathfrak{g} \otimes \mathfrak{g}$. Then

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = \frac{1}{2}[r_{13} + r_{31}, r_{23} + r_{32}] \tag{45}$$

is called the **type II Classical Yang-Baxter Equation (type II CYBE)**.

The following conclusion follows directly from Theorem 3.9 for $\kappa = 1$.

Proposition 3.12. *Let $\mathfrak{g}$ be a Lie algebra and $r \in \mathfrak{g}\otimes\mathfrak{g}$. Let $\alpha, \beta : \mathfrak{g}^* \to \mathfrak{g}$ be two linear maps given by Eq. (35). Suppose that β, regarded as an element of $\mathfrak{g} \otimes \mathfrak{g}$, is invariant. Then r is a solution of type II CYBE if and only if α is an extended $\mathcal{O}$-operator with extention β of mass 1, i.e., Eq. (44) holds. In this case, $(\mathfrak{g}, r)$ becomes a coboundary Lie bialgebra.*

Corollary 3.13. *Let $\mathfrak{g}$ be a Lie algebra and $r \in \mathfrak{g} \otimes \mathfrak{g}$. Let $\alpha, \beta : \mathfrak{g}^* \to \mathfrak{g}$ be the two linear maps given by Eq. (35). Suppose that β, regarded as an element of $\mathfrak{g} \otimes \mathfrak{g}$, is invariant. Define $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$, where $i = \sqrt{-1}$, and regard $\hat{\mathfrak{g}}$ as a real Lie algebra. The following conditions are equivalent:*

(i) *r is a solution of the type II CYBE.*
(ii) *α is an extended $\mathcal{O}$-operator with extention β of mass 1.*
(iii) *Regarding α and $i\beta$ as linear maps from $\hat{\mathfrak{g}}^* = \mathfrak{g}^* \oplus i\mathfrak{g}^*$ to $\hat{\mathfrak{g}}$, α is an extended $\mathcal{O}$-operator with extention $i\beta$ of mass -1.*
(iv) *$\alpha \pm i\beta$ are solutions of the CYBE in $\hat{\mathfrak{g}}$.*
(v) *$\alpha \pm i\beta$, regarded as linear maps from $\hat{\mathfrak{g}}^* = \mathfrak{g}^* \oplus i\mathfrak{g}^*$ to $\hat{\mathfrak{g}}$, satisfy*

$$(\alpha \pm i\beta)([a^*, b^*]_\delta) = [(\alpha \pm i\beta)(a^*), (\alpha \pm i\beta)(b^*)], \quad \forall a^*, b^* \in \mathfrak{g}^* \subset \hat{\mathfrak{g}}^* = \mathfrak{g}^* \oplus i\mathfrak{g}^*, \tag{46}$$

where the Lie algebra structure $[\,,\,]_\delta$ on $\mathfrak{g}^*$ is given by Eq. (37).

Proof. By Proposition 3.12, we have (i)⇔(ii). It follows from the definition of extended $\mathcal{O}$-operators that (ii)⇔(iii). Moreover, applying Proposition 3.8 to $\hat{\mathfrak{g}}$, we have (iii)⇔(iv). To prove (iv)⇔(v), we note that Proposition 3.8 also gives the equivalence of (iv) with the equation

$$(\alpha \pm i\beta)([u, v]_\delta) = [(\alpha \pm i\beta)(u), (\alpha \pm i\beta)(v)], \quad \forall u, v \in \hat{\mathfrak{g}}^* = \mathfrak{g}^* \oplus i\mathfrak{g}^*, \tag{47}$$

where

$$[u, v]_\delta = \mathrm{ad}^*(\alpha(u))v - \mathrm{ad}^*(\alpha(v))u, \quad \forall u, v \in \hat{\mathfrak{g}}^* = \mathfrak{g}^* \oplus i\mathfrak{g}^*.$$

Then (iv)⇔(v) follows since Eq. (47) ⇔ Eq. (46) by the definition of extended $\mathcal{O}$-operators. □

3.4. **Type II quasitriangular Lie bialgebras.** Considering the important role played by the Manin triple and Drinfeld's double from a Lie bialgebra in the classification of the Poisson homogeneous spaces and symplectic leaves of the corresponding Poisson-Lie groups [18, 25, 44, 52], it is important to investigate such Manin triple, as in [25, 33, 47]. However, explicit structures for Manin triples have been obtained only in special cases, such as for quasitriangular Lie bialgebras in [25]. Making use of the relationship between type II CYBE and extended $\mathcal{O}$-operators as displayed in Proposition 3.12, we consider the following class of Lie bialgebras and obtain a similar explicit constructions of their Manin triples.

Definition 3.14. A coboundary Lie bialgebra $(\mathfrak{g}, r)$ is said to be **type II quasitriangular** if it arises from a solution r of type II CYBE given by Eq. (45).

Our strategy is to express the Drinfeld's double $\mathcal{D}(\mathfrak{g})$ as an extension of a Lie algebra by an abelian Lie algebra, both derived from the extended $\mathcal{O}$-operator associated to the solution r of the type II CYBE. We then obtain the structure of the Manin triple explicitly in terms of this extension.

3.4.1. *An Lie algebra extension associated to a type II quasitriangular Lie bialgebra.* We obtain the Lie algebra extension from a type II quasitriangular Lie bialgebra by an exact sequence. Let $\mathfrak{g}$ be a Lie algebra and $r \in \mathfrak{g} \otimes \mathfrak{g}$. Define the symmetric and skew-symmetric parts α and β by Eq. (35).

Lemma 3.15. *With the same conditions as above, suppose that* $(\mathfrak{g}, r)$ *is a Lie bialgebra and* β *is invariant.*

(i) *For any* $x \in \mathfrak{g}, a^* \in \mathfrak{g}^*$, *we have*

$$\mathrm{ad}^*(a^*)x = -[x, \alpha(a^*)] + \alpha(\mathrm{ad}^*(x)a^*).$$

(ii) *If r is a solution of type II CYBE, then*

$$[(-\alpha(a^*), a^*), (-\alpha(b^*), b^*)]_{\mathcal{D}(\mathfrak{g})} = (-[\beta(a^*), \beta(b^*)], 0), \quad \forall a^*, b^* \in \mathfrak{g}^*.$$

Proof. (i) By Lemma 3.7, for any $x \in \mathfrak{g}, a^*, b^* \in \mathfrak{g}^*$, we have

$$\begin{aligned}\langle \mathrm{ad}^*(a^*)x, b^*\rangle &= \langle x, [b^*, a^*]_\delta\rangle = \langle x, -\mathrm{ad}^*(\alpha(a^*))b^* + \mathrm{ad}^*(\alpha(b^*))a^*\rangle \\ &= \langle -[x, \alpha(a^*)] + \alpha(\mathrm{ad}^*(x)a^*), b^*\rangle,\end{aligned}$$

where the last equality follows from the fact that α is skew-symmetric.

(ii) Since r is a solution of type II CYBE and β is invariant, by Proposition 3.12, α and β satisfy Eq. (44). So by Lemma 3.7 and Item (i), for any $a^*, b^* \in \mathfrak{g}^*$ we have

$$\begin{aligned}&[(-\alpha(a^*), a^*), (-\alpha(b^*), b^*)]_{\mathcal{D}(\mathfrak{g})} \\ =\ & ([\alpha(a^*), \alpha(b^*)] - \mathrm{ad}^*(a^*)\alpha(b^*) + \mathrm{ad}^*(b^*)\alpha(a^*), [a^*, b^*]_\delta - \mathrm{ad}^*(\alpha(a^*))b^* + \mathrm{ad}^*(\alpha(b^*))a^*) \\ =\ & ([\alpha(a^*), \alpha(b^*)] + [\alpha(b^*), \alpha(a^*)] - \alpha(\mathrm{ad}^*(\alpha(b^*))a^*) - [\alpha(a^*), \alpha(b^*)] + \alpha(\mathrm{ad}^*(\alpha(a^*))b^*), 0) \\ =\ & (-[\alpha(a^*), \alpha(b^*)] + \alpha(\mathrm{ad}^*(\alpha(a^*))b^* - \mathrm{ad}^*(\alpha(b^*))a^*), 0) = (-[\beta(a^*), \beta(b^*)], 0).\end{aligned}$$

□

Now let $(\mathfrak{g}, r)$ be a type II quasitriangular Lie bialgebra. By Proposition 3.3, $\beta \in \mathfrak{g} \otimes \mathfrak{g}$ is invariant. Regarding β as a linear map from $\mathfrak{g}^*$ to $\mathfrak{g}$, we define

$$\mathfrak{k} = \mathrm{Im}\beta, \quad \mathfrak{k}^\perp = \mathrm{Ker}\beta.$$

Then by Lemma 3.6, $\mathfrak{k}$ is an ideal of $\mathfrak{g}$. On the other hand, define $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$, where $i = \sqrt{-1}$, and regard $\hat{\mathfrak{g}}$ as a real Lie algebra. Let $\mathcal{D}(\mathfrak{g}) \equiv \mathfrak{g} \oplus \mathfrak{g}^*$ be the Drinfeld's double defined in Proposition 3.2.

Proposition 3.16. *With the notations explained above, define two linear maps* $\Theta_\pm : \mathcal{D}(\mathfrak{g}) \to \hat{\mathfrak{g}}$ *by*

$$\Theta_\pm(x, a^*) = x + \alpha(a^*) \pm i\beta(a^*), \quad \forall x \in \mathfrak{g}, a^* \in \mathfrak{g}^*. \tag{48}$$

Then $\Theta_\pm$ *are homomorphisms of Lie algebras. Moreover,* $\mathrm{Ker}\Theta_+ = \mathrm{Ker}\Theta_-$ *is an abelian Lie subalgebra of* $\mathcal{D}(\mathfrak{g})$.

Proof. First, it is obvious that for any $x, y \in \mathfrak{g}$,

$$\Theta_\pm([x, y]_{\mathcal{D}(\mathfrak{g})}) = [\Theta_\pm(x), \Theta_\pm(y)]_{\hat{\mathfrak{g}}}.$$

On the other hand, by Corollary 3.13.(v), Eq. (46) holds, that is, for any $a^*, b^* \in \mathfrak{g}^*$, we have

$$\Theta_\pm([a^*, b^*]_{\mathcal{D}(\mathfrak{g})}) = [\Theta_\pm(a^*), \Theta_\pm(b^*)]_{\hat{\mathfrak{g}}}.$$

Furthermore, by Lemma 3.6 and Lemma 3.15.(i), we have

$$\begin{aligned}\Theta_\pm([x, a^*]_{\mathcal{D}(\mathfrak{g})}) &= \Theta_\pm(\mathrm{ad}^*(x)a^* - \mathrm{ad}^*(a^*)x) = \alpha(\mathrm{ad}^*(x)a^*) - \mathrm{ad}^*(a^*)x \pm i\beta(\mathrm{ad}^*(x)a^*)\\ &= [x, \alpha(a^*)] \pm i[x, \beta(a^*)] = [x, (\alpha \pm i\beta)(a^*)]_{\hat{\mathfrak{g}}} = [\Theta_\pm(x), \Theta_\pm(a^*)]_{\hat{\mathfrak{g}}}.\end{aligned}$$

So $\Theta_\pm$ are homomorphisms of Lie algebras.

Moreover, it is easy to show that

$$\mathrm{Ker}\Theta_+ = \mathrm{Ker}\Theta_- = \{(-\alpha(a^*), a^*) | a^* \in \mathfrak{k}^\perp = \mathrm{Ker}\beta\}.$$

By Lemma 3.15.(ii), for any $a^*, b^* \in \mathfrak{k}^\perp = \mathrm{Ker}\beta$, we have

$$[(-\alpha(a^*), a^*), (-\alpha(b^*), b^*)]_{\mathcal{D}(\mathfrak{g})} = (-[\beta(a^*), \beta(b^*)], 0) = (0, 0).$$

So $\mathrm{Ker}\Theta_+ = \mathrm{Ker}\Theta_-$ is an abelian Lie subalgebra of $\mathcal{D}(\mathfrak{g})$. □

Equip the space $\mathfrak{k}^\perp = \mathrm{Ker}\beta$ with the structure of an abelian Lie algebra. Define a linear map $\iota : \mathfrak{k}^\perp \to \mathcal{D}(\mathfrak{g})$ by

$$\iota(a^*) = (-\alpha(a^*), a^*), \quad \forall a^* \in \mathfrak{k}^\perp.$$

Then ι is in fact an embedding of Lie algebras whose image coincides with $\mathrm{Ker}\Theta_+ = \mathrm{Ker}\Theta_-$. On the other hand, the images of $\Theta_\pm$ in $\hat{\mathfrak{g}} = \mathfrak{g} \oplus i\mathfrak{g}$ are $\mathfrak{g} \oplus i\mathrm{Im}\beta = \mathfrak{g} \oplus i\mathfrak{k}$, which is a Lie subalgebra of $\hat{\mathfrak{g}}$. Thus we have

Proposition 3.17. *The sequences*

$$0 \longrightarrow \mathfrak{k}^\perp \xrightarrow{\iota} \mathcal{D}(\mathfrak{g}) \xrightarrow{\Theta_\pm} \mathfrak{g} \oplus i\mathfrak{k} \longrightarrow 0 \tag{49}$$

are exact.

As a special case, we have

Corollary 3.18. ([34]) *Let* $\mathfrak{g}$ *be a Lie algebra and* $r \in \mathfrak{g} \otimes \mathfrak{g}$. *Define* α *and* β *by Eq. (35). Suppose that* β *is invariant and invertible (regarded as a linear map from* $\mathfrak{g}^*$ *to* $\mathfrak{g}$*). If* $(\mathfrak{g}, r)$ *is a type II quasitriangular Lie bialgebra, then* $\Theta_\pm : \mathcal{D}(\mathfrak{g}) \to \mathfrak{g} \oplus i\mathfrak{g}$ *are isomorphisms of Lie algebras.*

Proof. In this case, $\mathrm{Ker}\Theta_+ = \mathrm{Ker}\Theta_- = 0$ and $\mathrm{Im}\Theta_+ = \mathrm{Im}\Theta_- = \mathfrak{g} \oplus i\mathfrak{g}$. □

3.4.2. *Description of the extension.* According to Proposition 3.17, $\mathcal{D}(\mathfrak{g})$ is an extension of $\mathfrak{g}\oplus i\mathfrak{k}$ by the abelian Lie algebra $\mathfrak{k}^\perp$. So there is an induced representation of $\mathfrak{g}\oplus i\mathfrak{k}$ on $\mathfrak{k}^\perp$ and the extension is uniquely defined by an element of $H^2(\mathfrak{g}\oplus i\mathfrak{k}, \mathfrak{k}^\perp)$. To describe these structures explicitly, we need to fix two splittings $S_\pm : \mathfrak{g}\oplus i\mathfrak{k} \to \mathcal{D}(\mathfrak{g})$ of Eq. (49) in the category of vector spaces, that is, $\Theta_\pm \circ S_\pm = \mathrm{id}_{\mathfrak{g}\oplus i\mathfrak{k}}$ such that $S(0) = 0$. In fact, suppose that $s : \mathfrak{k} \to \mathfrak{g}^*$ is a right inverse of $\beta : \mathfrak{g}^* \to \mathfrak{k} \subset \mathfrak{g}$, that is, $\beta\circ s = \mathrm{id}_{\mathfrak{k}}$, then the desired splittings $S_\pm : \mathfrak{g}\oplus i\mathfrak{k} \to \mathcal{D}(\mathfrak{g})$ are defined by

$$S_\pm(x+iy) = x \mp \alpha s(y) \pm s(y), \quad \forall x\in\mathfrak{g}, y\in\mathfrak{k}.$$

Recall that the construction of a Lie algebra $\mathfrak{h}$ by a $\mathfrak{h}$-module V associated to a cohomology class $[\tau]\in H^2(\mathfrak{h}, V)$ is the vector space $\mathfrak{h}\oplus V$ equipped with the bracket $[(x,u),(y,v)] = ([x,y], x\cdot v - y\cdot u + \tau(x,y)), \quad \forall x,y\in\mathfrak{h}, u,v\in V$. We denote such extension by $\mathfrak{h}\ltimes_\tau V$.

Returning to $\mathcal{D}(\mathfrak{g})$, we shall write down the actions of $\mathfrak{g}\oplus i\mathfrak{k}$ on $\mathfrak{k}^\perp$ and the cohomology classes $\tau_\pm$ explicitly.

Lemma 3.19. *The actions of $\mathfrak{g}\oplus i\mathfrak{k}$ on $\mathfrak{k}^\perp$ induced from the extensions defined by Eq. (49) are given by $(x+iy)\cdot_\pm a^* = \mathrm{ad}^*(x)a^*$, for any $x\in\mathfrak{g}, y\in\mathfrak{k}, a^*\in\mathfrak{k}^\perp$.*

Proof. According to Lemma 3.15, for any $x\in\mathfrak{g}, y\in\mathfrak{k}, a^*\in\mathfrak{k}^\perp$, we have

$$\begin{aligned}
[S_\pm(x+iy), \iota(a^*)] &= [x\mp\alpha(s(y))\pm s(y), -\alpha(a^*)+a^*] \\
&= [x, -\alpha(a^*)+a^*] \pm [\beta(s(y)), \beta(a^*)] \\
&= -[x,\alpha(a^*)] - \mathrm{ad}^*(a^*)x + \mathrm{ad}(x)a^* = \iota(\mathrm{ad}(x)a^*).
\end{aligned}$$

So the actions are given by $(x+iy)\cdot_\pm a^* = \iota^{-1}([S(x+iy), \iota(a^*)]) = \mathrm{ad}^*(x)a^*$. □

Theorem 3.20. *Define two forms $\tau_\pm : (\mathfrak{g}\oplus i\mathfrak{k})\otimes(\mathfrak{g}\oplus i\mathfrak{k}) \to \mathfrak{k}^\perp$ by*

$$\tau_\pm(x_1+iy_1, x_2+iy_2) = \pm(\mathrm{ad}^*(x_1)s(y_2) - \mathrm{ad}^*(x_2)s(y_1) - s([x_1,y_2]) + s([x_2,y_1])),$$

for any $x_1,x_2\in\mathfrak{g}, y_1,y_2\in\mathfrak{k}^\perp$. Then the forms $\tau_\pm$ are 2-cocycles and

$$\mathcal{D}(\mathfrak{g}) \cong (\mathfrak{g}\oplus i\mathfrak{k})\ltimes_{\tau_\pm}\mathfrak{k}^\perp. \tag{50}$$

Proof. The cohomology classes associated to the extensions defined by Eq. (49) are the classes of the 2-cocycles $(x_1,x_2\in\mathfrak{g}, y_1,y_2\in\mathfrak{k}^\perp)$

$$\begin{aligned}
&\iota^{-1}([S_\pm(x_1+iy_1), S_\pm(x_2+iy_2)] - S_\pm([x_1+iy_1, x_2+iy_2])) \\
=\ &\iota^{-1}([x_1\mp\alpha(s(y_1))\pm s(y_1), x_2\mp\alpha(s(y_2))\pm s(y_2)] - S_\pm([x_1,x_2]-[y_1,y_2]+i([x_1,y_2]+[y_1,x_2])) \\
=\ &\iota^{-1}([x_1,x_2] + [x_1, \pm(-\alpha(s(y_2))+s(y_2))] + [\pm(-\alpha(s(y_1))+s(y_1)), x_2] + [-\alpha(s(y_1)) + \\
&s(y_1), -\alpha(s(y_2))+s(y_2)] - [x_1,x_2] + [y_1,y_2] \pm \alpha(s([x_1,y_2]+[y_1,x_2])) \mp s([x_1,y_2]+[y_1,x_2])) \\
=\ &\iota^{-1}(\pm\iota(\mathrm{ad}^*(x_1)(s(y_2))) \mp \iota(\mathrm{ad}^*(x_2)(s(y_1))) - [\beta(s(y_1)), \beta(s(y_2))] + [y_1,y_2] \\
&\mp\iota(s([x_1,y_2]+[y_1,x_2]))) \\
=\ &\pm(\mathrm{ad}^*(x_1)s(y_2) - \mathrm{ad}^*(x_2)s(y_1) - s([x_1,y_2]) + s([x_2,y_1])),
\end{aligned}$$

where the third equality follows from Lemma 3.15. □

3.4.3. *The embeddings of $\mathfrak{g}$ and $\mathfrak{g}^*$ in $\mathcal{D}(\mathfrak{g})$ and the description of the Manin triple.* We now apply the isomorphisms in Eq. (50) to describe the structure of the Manin triple $(\mathcal{D}(\mathfrak{g}), \mathfrak{g}, \mathfrak{g}^*)$ explicitly in terms of $(\mathfrak{g}\oplus i\mathfrak{k})\ltimes_{\tau_\pm}\mathfrak{k}^\perp$.

It is clear that from the identifications defined by Eq. (50), $\mathfrak{g}$ is embedded in $\mathcal{D}(\mathfrak{g})$ by

$$x \mapsto (x,0)\ltimes_{\tau_\pm} 0 \in (\mathfrak{g}\oplus i\mathfrak{k})\ltimes_{\tau_\pm}\mathfrak{k}^\perp, \quad \forall x\in\mathfrak{g}. \tag{51}$$

Moreover, for any $a^* \in \mathfrak{g}^*$, we have $a^* - s(\beta(a^*)) \in \mathfrak{f}^\perp$ and $a^* = S_\pm(\alpha(a^*) \pm i\beta(a^*)) + \iota(a^* - s(\beta(a^*)))$. So the embeddings of $\mathfrak{g}^*$ in $(\mathfrak{g} \oplus i\mathfrak{f}) \ltimes_{\tau_\pm} \mathfrak{f}^\perp \cong \mathcal{D}(\mathfrak{g})$ are given by

$$a^* \mapsto (\alpha(a^*) \pm i\beta(a^*)) \ltimes_{\tau_\pm} (a^* - s(\beta(a^*))). \tag{52}$$

To describe the embeddings of $\mathfrak{g}^*$ more explicitly, we first recall some results in [25] about classification of subalgebras of extensions of the form $\mathfrak{h} \ltimes_\tau V$, where $\mathfrak{h}$ is a Lie algebra, V is an $\mathfrak{h}$-module and $\tau \in H^2(\mathfrak{h}, V)$. Let $p : \mathfrak{h} \ltimes_\tau V \to \mathfrak{h}$ and $q : \mathfrak{h} \ltimes_\tau V \to V$ be the projections $p(h,u) = h$ and $q(h,u) = u$ for any $h \in \mathfrak{h}, u \in V$.

Theorem 3.21. ([25]) *Let $\mathfrak{b}$ be a Lie subalgebra of $\mathfrak{h}$ and W be a $\mathfrak{b}$-submodule of V. Let $\phi : \mathfrak{b} \to V/W$ be a 1-cochain whose coboundary is $-\epsilon \circ \tau|_{\mathfrak{b}}$, where ϵ denotes the projection $V \to V/W$. Define*

$$\mathfrak{b}_W^\phi = \{(x,u) | x \in \mathfrak{b}, u + W = \phi(x)\}.$$

Then $\mathfrak{b}_W^\phi$ is a Lie subalgebra of $\mathfrak{h} \ltimes_\tau V$. Conversely, if $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{h} \ltimes_\tau V$, then $\mathfrak{k}$ is of the form $\mathfrak{b}_W^\phi$, where $\mathfrak{b} = p(\mathfrak{k}), W = \mathfrak{k} \cap V$ and $\phi : \mathfrak{b} \to V/W$ is given by $\phi(x) = q(p^{-1}(x)) + W$, for any $x \in \mathfrak{b}$.

We now identify $\mathfrak{g}^*$ with its embedded images inside $(\mathfrak{g} \oplus i\mathfrak{f}) \ltimes_{\tau_\pm} \mathfrak{f}^\perp$. It follows from Eq. (52) that $W = \mathrm{Ker}\alpha \cap \mathrm{Ker}\beta$ and $\mathfrak{b}_\pm = \Theta_\pm(\mathfrak{g}^*) = \{\alpha(a^*) \pm i\beta(a^*) | a^* \in \mathfrak{g}^*\}$, where $\Theta_\pm$ are defined by Eq. (48). Furthermore the projections $p_\pm|_{\mathfrak{g}^*} : \mathfrak{g}^* \to \Theta_\pm(\mathfrak{g}^*)$ factor through the isomorphisms $\overline{p}_\pm : \mathfrak{g}^*/W \to \mathfrak{b}_\pm$ given by

$$\overline{p}_\pm(a^* + W) = \alpha(a^*) \pm i\beta(a^*), \quad \forall a^* \in \mathfrak{g}^*,$$

respectively. Hence the 1-cochains $\phi_\pm : \mathfrak{b}_\pm \to \overline{\mathfrak{f}^\perp} = \mathfrak{f}^\perp/W$ of Theorem 3.21 in this situation are given by

$$\begin{aligned} \phi_\pm(x+iy) &= \overline{p}_\pm^{-1}(x+iy) - \epsilon s\beta\overline{p}_\pm^{-1}(x+iy) \\ &= \overline{p}_\pm^{-1}(x+iy) \mp \epsilon s(y). \end{aligned} \tag{53}$$

Thus we have

Theorem 3.22. *The images of $\mathfrak{g}^*$ inside $\mathcal{D}(\mathfrak{g})$ under the isomorphisms $\mathcal{D}(\mathfrak{g}) \cong (\mathfrak{g} \oplus i\mathfrak{f}) \ltimes_{\tau_\pm} \mathfrak{f}^\perp$ coincide with the subalgebras $\mathfrak{b}_{\pm W}^{\phi_\pm}$ respectively, where $\mathfrak{b}_\pm = \Theta_\pm(\mathfrak{g}^*)$, $W = \mathrm{Ker}\alpha \cap \mathrm{Ker}\beta$ and $\phi_\pm : \mathfrak{b}_\pm \to \overline{\mathfrak{f}^\perp}$ are described by Eq. (53).*

Remark 3.23. One can define a **type II quasitriangular Poisson-Lie group** as a simply connected Poisson-Lie group whose tangent Lie bialgebra is a type II quasitriangular Lie bialgebra. Moreover, one can investigate the above descriptions of the structure of $\mathcal{D}(\mathfrak{g})$ and the embeddings of $\mathfrak{g}$ and $\mathfrak{g}^*$ in $\mathcal{D}(\mathfrak{g})$ in the context of (type II quasitriangular) Poisson-Lie groups. For the corresponding discussion of quasitriangular Lie bialgebras and quasitriangular Poisson-Lie groups, see the study in [25].

We end our explicit description of the Manin triple $(\mathcal{D}(\mathfrak{g}), \mathfrak{g}, \mathfrak{g}^*)$ in terms of the isomorphisms in Eq. (50) by expressing the bilinear form $\mathfrak{B}_p$ in Eq. (29). For any

$$d = x + iy \ltimes_{\tau_\pm} \eta \in (\mathfrak{g} \oplus i\mathfrak{f}) \ltimes_{\tau_\pm} \mathfrak{f}^\perp, \quad x \in \mathfrak{g}, y \in \mathfrak{f}, \eta \in \mathfrak{f}^\perp,$$

define

$$\Xi_\pm(d) \equiv x - \alpha(\eta) \mp \alpha(s(y)) \in \mathfrak{g}, \quad \Lambda_\pm(d) \equiv \eta \pm s(y) \in \mathfrak{g}^*.$$

Using Eq. (51) and Eq. (52), it is obvious that the compositions of the isomorphisms $(\mathfrak{g} \oplus i\mathfrak{f}) \ltimes_{\tau_\pm} \mathfrak{f}^\perp \cong \mathcal{D}(\mathfrak{g}) \cong \mathfrak{g} \oplus \mathfrak{g}^*$ are given by $d \mapsto (\Xi_\pm(d), \Lambda_\pm(d))$ respectively. Therefore, the bilinear forms given by Eq. (29) on $(\mathfrak{g} \oplus i\mathfrak{f}) \ltimes_{\tau_\pm} \mathfrak{f}^\perp \cong \mathcal{D}(\mathfrak{g})$ satisfy

$$\mathfrak{B}_\pm(d_1, d_2) \equiv \langle \Lambda_\pm(d_1), \Xi_\pm(d_2) \rangle + \langle \Lambda_\pm(d_2), \Xi_\pm(d_1) \rangle.$$

4. Self-dual Lie algebras and factorizable (type II) quasitriangular Lie bialgebras

We will focus on extended $\mathcal{O}$-operators on self-dual Lie algebras and the related (type II) factorizable quasitriangular Lie bialgebras in this section. We first obtain finer properties of the various extended $\mathcal{O}$-operators (in Eq. (22) and Eq. (38)) and the ECYBE in this context. We then apply these properties to provide new examples of (type II) factorizable quasitriangular Lie bialgebras.

4.1. Extended $\mathcal{O}$-operators and the ECYBE on self-dual Lie algebras.

Definition 4.1. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{B}:\mathfrak{g}\otimes\mathfrak{g}\to\mathbb{R}$ be a bilinear form. Suppose that $R:\mathfrak{g}\to\mathfrak{g}$ is a linear endomorphism of $\mathfrak{g}$. Then R is called **self-adjoint** (resp. **skew-adjoint**) with respect to $\mathfrak{B}$ if

$$\mathfrak{B}(R(x),y)=\mathfrak{B}(x,R(y))\quad (\text{resp. } \mathfrak{B}(R(x),y)=-\mathfrak{B}(x,R(y)))$$

for any $x,y\in\mathfrak{g}$.

Lemma 4.2. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{B}:\mathfrak{g}\otimes\mathfrak{g}\to\mathbb{R}$ be a nondegenerate symmetric invariant bilinear form. Let $\varphi:\mathfrak{g}\to\mathfrak{g}^*$ be defined from $\mathfrak{B}$ by Eq. (8). Suppose that $\beta:\mathfrak{g}\to\mathfrak{g}$ is an endomorphism that is self-adjoint with respect to $\mathfrak{B}$. Then for a given $\kappa\in\mathbb{R}$, β is antisymmetric of mass κ and $\mathfrak{g}$-invariant of mass κ, i.e., it satisfies Eq. (21), if and only if $\tilde{\beta}=\beta\varphi^{-1}:\mathfrak{g}^*\to\mathfrak{g}$ is antisymmetric of mass κ and $\mathfrak{g}$-invariant of mass κ, i.e.,*

$$\kappa\tilde{\beta}(\mathrm{ad}^*(x)a^*)=\kappa[x,\tilde{\beta}(a^*)],\quad \forall x\in\mathfrak{g}, a^*\in\mathfrak{g}^*, \tag{54}$$

$$\kappa\,\mathrm{ad}^*(\tilde{\beta}(a^*))b^*+\kappa\,\mathrm{ad}^*(\tilde{\beta}(b^*))a^*=0,\quad \forall a^*,b^*\in\mathfrak{g}^*. \tag{55}$$

Proof. When $\kappa=0$, the conclusion is obvious. Now we assume $\kappa\neq 0$. Since $\mathfrak{B}$ is symmetric and β is self-adjoint with respect to $\mathfrak{B}$, for any $a^*,b^*\in\mathfrak{g}^*$ and $x=\varphi^{-1}(a^*), y=\varphi^{-1}(b^*)\in\mathfrak{g}$, we have $\langle\beta(x),\varphi(y)\rangle=\langle\varphi(x),\beta(y)\rangle$. Hence $\langle\tilde{\beta}(a^*),b^*\rangle=\langle a^*,\tilde{\beta}(b^*)\rangle$, that is, $\tilde{\beta}$ as an element of $\mathfrak{g}\otimes\mathfrak{g}$ is symmetric. So by Lemma 3.6, Eq. (54) and Eq. (55) are equivalent. On the other hand, since $\mathfrak{B}$ is symmetric and invariant and β is self-adjoint with respect to $\mathfrak{B}$, for any $z\in\mathfrak{g}$, we have

$$\begin{aligned}\langle\mathrm{ad}^*(\tilde{\beta}(a^*))b^*,z\rangle &= \langle b^*,[z,\beta(x)]\rangle=\mathfrak{B}(y,[z,\beta(x)])=\mathfrak{B}([y,z],\beta(x))=\mathfrak{B}(\beta([y,z]),x),\\ \langle\mathrm{ad}^*(\tilde{\beta}(b^*))a^*,z\rangle &= \mathfrak{B}(x,[z,\beta(y)]).\end{aligned}$$

Since $\mathfrak{B}$ is nondegenerate, $\mathrm{ad}^*(\tilde{\beta}(a^*))b^*+\mathrm{ad}^*(\tilde{\beta}(b^*))a^*=0$ if and only if $\beta([y,z])=[\beta(y),z]$, which is equivalent to the fact that β satisfies Eq. (21) for $k\neq 0$. So the conclusion follows. $\square$

Proposition 4.3. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{B}:\mathfrak{g}\otimes\mathfrak{g}\to\mathbb{R}$ be a nondegenerate symmetric invariant bilinear form. Let $\varphi:\mathfrak{g}\to\mathfrak{g}^*$ be defined from $\mathfrak{B}$ by Eq. (8). Suppose that R and β are two linear endomorphisms of $\mathfrak{g}$ and β is self-adjoint with respect to $\mathfrak{B}$. Let $\kappa\in\mathbb{R}$ be given.*

(i) *R is an extended $\mathcal{O}$-operator with extention β of mass κ, i.e., β satisfies Eq. (21) and R and β satisfy Eq. (22), if and only if $\tilde{R}=R\varphi^{-1}:\mathfrak{g}^*\to\mathfrak{g}$ is an extended $\mathcal{O}$-operator with extention $\tilde{\beta}=\beta\varphi^{-1}:\mathfrak{g}^*\to\mathfrak{g}$ of mass κ, i.e., $\tilde{\beta}$ satisfies Eq. (54) and Eq. (55) and $\tilde{R}$ and $\tilde{\beta}$ satisfy Eq. (38) for $\alpha=\tilde{R}$ and $\beta=\tilde{\beta}$, where the linear map $\varphi:\mathfrak{g}\to\mathfrak{g}^*$ is defined by Eq. (8).*

(ii) *Suppose in addition that R is skew-adjoint with respect to $\mathfrak{B}$. Then $r_\pm=\tilde{R}\pm\tilde{\beta}$ regarded as an element of $\mathfrak{g}\otimes\mathfrak{g}$ is a solution of ECYBE of mass $\frac{\kappa+1}{4}$ if and only if R is an extended $\mathcal{O}$-operator with extention β of mass κ.*

Proof. (i) First, by Lemma 4.2 we know that β is antisymmetric of mass κ and $\mathfrak{g}$-invariant of mass κ if and only if $\tilde{\beta} = \beta\varphi^{-1}$ is antisymmetric of mass κ and $\mathfrak{g}$-invariant of mass κ. On the other hand, since $\mathfrak{B}$ is symmetric and invariant, for any $x, y, z \in \mathfrak{g}$, we have

$$\mathfrak{B}([x,y],z) = \mathfrak{B}(x,[y,z]) \Leftrightarrow \langle\varphi([x,y]),z\rangle = \langle\varphi(x),[y,z]\rangle \Leftrightarrow \varphi(\mathrm{ad}(y)x) = \mathrm{ad}^*(y)\varphi(x). \tag{56}$$

For any $x, y \in \mathfrak{g}$, put $a^* = \varphi(x), b^* = \varphi(y)$. Since φ is invertible, Eq. (22) can be written as

$$[\tilde{R}(a^*),\tilde{R}(b^*)] - \tilde{R}(\varphi([\tilde{R}(a^*),\varphi^{-1}(b^*)] + [\varphi^{-1}(a^*),\tilde{R}(b^*)])) = k[\tilde{\beta}(a^*),\tilde{\beta}(b^*)].$$

By Eq. (56), the above equation is equivalent to

$$[\tilde{R}(a^*),\tilde{R}(b^*)] - \tilde{R}(\mathrm{ad}^*(\tilde{R}(a^*))b^* - \mathrm{ad}^*(\tilde{R}(b^*))a^*) = \kappa[\tilde{\beta}(a^*),\tilde{\beta}(b^*)].$$

So R is an extended $\mathcal{O}$-operator with extention β of mass κ if and only if $\tilde{R} = R\varphi^{-1} : \mathfrak{g}^* \to \mathfrak{g}$ is an extended $\mathcal{O}$-operator with extention $\tilde{\beta}$ of mass κ.

(ii) Furthermore, if R is skew-adjoint with respect to $\mathfrak{B}$, then $\langle R(x),\varphi(y)\rangle + \langle\varphi(x),R(y)\rangle = 0$. Hence $\langle\tilde{R}(a^*),b^*\rangle + \langle a^*,\tilde{R}(b^*)\rangle = 0$, that is, $\tilde{R}$ regarded as an element of $\mathfrak{g}\otimes\mathfrak{g}$ is skew-symmetric. Therefore, the conclusion (ii) follows from Item (i) and Theorem 3.9. □

As special cases of Proposition 4.3.(ii), we have

Corollary 4.4. *Under the same assumptions as in Proposition 4.3.(ii), we have*

(i) *If $\kappa = -1$, then $r_\pm = \tilde{R}\pm\tilde{\beta}$ as an element of $\mathfrak{g}\otimes\mathfrak{g}$ is a solution of the CYBE (Eq. (32)), namely $(\mathfrak{g}, r_\pm)$ is a quasitriangular Lie bialgebra, if and only if R is an extended $\mathcal{O}$-operator with extention β of mass -1, that is, β satisfies Eq. (21) for $\kappa \neq 0$ and R and β satisfy Eq. (22) for $\kappa = -1$.*

(ii) *If $\kappa = 1$, then $r_\pm = \tilde{R}\pm\tilde{\beta}$ as an element of $\mathfrak{g}\otimes\mathfrak{g}$ is a solution of type II CYBE (Eq. (45)), namely $(\mathfrak{g}, r_\pm)$ is a type II quasitriangular Lie bialgebra, if and only if R is an extended $\mathcal{O}$-operator with extention β of mass 1, that is, β satisfies Eq. (21) for $\kappa \neq 0$ and R and β satisfy Eq. (22) for $\kappa = 1$.*

Remark 4.5. Conclusion (i) in the above corollary in the special case when $\beta = \mathrm{id}_{\mathfrak{g}}$ can also be found in [28].

4.2. **Factorizable quasitriangular Lie bialgebras.** Recall that a quasitriangular Lie bialgebra $(\mathfrak{g}, r)$ is said to be **factorizable** if the symmetric part of r regarded as a linear map from $\mathfrak{g}^*$ to $\mathfrak{g}$ is invertible. Factorizable quasitriangular Lie bialgebras are related to the factorization problem in integrable systems [39]. Next we will provide some new examples of factorizable quasitriangular Lie bialgebras.

Lemma 4.6. *Let G be a simply connected Lie group whose Lie algebra is $\mathfrak{g}$. Let N be a linear transformation of $\mathfrak{g}$ which induces a left invariant $(1,1)$ tensor field on G. If there exists a left invariant torsion-free connection ∇ on G such that N is parallel with respect to ∇, then N is a Nijenhuis tensor, that is, it satisfies Eq. (24).*

Proof. Since N is parallel with respect to ∇, for any $x, y \in \mathfrak{g}$, we have that $N(\nabla_{\hat{x}}\hat{y}(e)) = \nabla_{\hat{x}}N(y)^\wedge(e)$, where $\hat{x}, \hat{y}$ are the left invariant vector fields generated by $x, y \in \mathfrak{g}$ respectively and e is the identity element of G. Moreover, since ∇ is torsion-free, for any $x, y \in \mathfrak{g}$, we show that

$$\begin{aligned}
[N(x),N(y)] + N^2([x,y]) &= \nabla_{N(x)^\wedge}N(y)^\wedge(e) - \nabla_{N(y)^\wedge}N(x)^\wedge(e) + N^2(\nabla_{\hat{x}}\hat{y}(e)) - N^2(\nabla_{\hat{y}}\hat{x}(e))\\
&= N(\nabla_{N(x)^\wedge}\hat{y}(e)) - N(\nabla_{\hat{y}}N(x)^\wedge(e)) + N(\nabla_{\hat{x}}N(y)^\wedge(e)) - N(\nabla_{N(y)^\wedge}\hat{x}(e))\\
&= N([N(x),y] + [x,N(y)]).
\end{aligned}$$

□

Lemma 4.7. *Let $(\mathfrak{g}, r)$ be a triangular Lie bialgebra, that is, r is a skew-symmetric solution of CYBE. Suppose that r regarded as a linear map from $\mathfrak{g}^*$ to $\mathfrak{g}$ is invertible. Define a family of linear maps $N_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} : \mathcal{D}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^* \to \mathcal{D}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*$ by*

$$N_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(x, a^*) = (\lambda_1 r(a^*) + \lambda_2 x, \lambda_3 r^{-1}(x) + \lambda_4 a^*), \quad \forall x \in \mathfrak{g}, a^* \in \mathfrak{g}^*, \lambda_i \in \mathbb{R}, i = 1, 2, 3, 4. \tag{57}$$

Then $N_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}$ is skew-adjoint with respect to the bilinear form $\mathfrak{B}_p$ defined by Eq. (29) if and only if $\lambda_2 + \lambda_4 = 0$.

The lemma is interesting on its own right since the simply connected Lie group corresponding to the Lie algebra in the lemma is a symplectic Lie group ([13, 16, 17]).

Proof. In fact, for any $x, y \in \mathfrak{g}, a^*, b^* \in \mathfrak{g}$, we have

$$\begin{aligned}
& \mathfrak{B}_p(N_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(x, a^*), (y, b^*)) + \mathfrak{B}_p((x, a^*), N_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(y, b^*)) \\
= \; & \mathfrak{B}_p((\lambda_1 r(a^*) + \lambda_2 x, \lambda_3 r^{-1}(x) + \lambda_4 a^*), (y, b^*)) + \mathfrak{B}_p((x, a^*), (\lambda_1 r(b^*) + \lambda_2 y, \lambda_3 r^{-1}(y) + \lambda_4 b^*)) \\
= \; & \lambda_1 \langle r(a^*), b^* \rangle + \lambda_2 \langle x, b^* \rangle + \lambda_3 \langle r^{-1}(x), y \rangle + \lambda_4 \langle a^*, y \rangle + \lambda_3 \langle x, r^{-1}(y) \rangle + \lambda_4 \langle x, b^* \rangle + \lambda_1 \langle a^*, r(b^*) \rangle \\
& + \lambda_2 \langle a^*, y \rangle = (\lambda_2 + \lambda_4)(\langle x, b^* \rangle + \langle a^*, y \rangle),
\end{aligned}$$

where the last equality follows from r being skew-symmetric. So the conclusion follows. □

Lemma 4.8. *With the conditions and notations in Lemma 4.7, the linear operator $N_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}$ defined by Eq. (57) is a Nijenhuis tensor on $\mathcal{D}(\mathfrak{g})$, that is, it satisfies Eq. (24) on $\mathcal{D}(\mathfrak{g})$.*

Proof. Let $\mathcal{D}(G)$ be the corresponding simply connected double Lie group of the Drinfeld's double $\mathcal{D}(g)$, where G denotes the simply connected Poisson-Lie group of the Lie bialgebra $(\mathfrak{g}, r)$. Then it is easy to see that the following equation defines a left invariant torsion-free connection (in fact, according to [16], it is also flat) on $\mathcal{D}(G)$:

$$\nabla_{(x,a^*)^\wedge}(y, b^*)^\wedge(e) = (r(\mathrm{ad}^*(x) r^{-1}(y)) + \mathrm{ad}^*(a^*) y, \mathrm{ad}^*(r(a^*)) b^* + \mathrm{ad}^*(x) b^*), \quad \forall x, y \in \mathfrak{g}, a^*, b^* \in \mathfrak{g}^*,$$

where $(x, a^*)^\wedge, (y, b^*)^\wedge$ are the left invariant vector fields generated by $(x, a^*), (y, b^*) \in \mathcal{D}(g)$ respectively and e is the identity element of $\mathcal{D}(G)$. We only need to prove that the tensor $N_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}$ defined by Eq. (57) is parallel with respect to the above connection, since then $N_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}$ satisfies Eq. (24) on $\mathcal{D}(\mathfrak{g})$ by Lemma 4.6. Now by Lemma 3.7, Corollary 3.10 and Lemma 3.15.(i), for any $a^*, b^* \in \mathfrak{g}^*$,

$$\mathrm{ad}^*(a^*) r(b^*) = -[r(b^*), r(a^*)] + r(\mathrm{ad}^*(r(b^*)) a^*) = -r([b^*, a^*]_\delta) + r(\mathrm{ad}^*(r(b^*)) a^*) = r(\mathrm{ad}^*(r(a^*)) b^*). \tag{58}$$

Moreover, for any $x, y \in \mathfrak{g}$,

$$\begin{aligned}
& \nabla_{(x,a^*)^\wedge} N_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(y, b^*)^\wedge(e) = \nabla_{(x,a^*)^\wedge}(\lambda_1 r(b^*) + \lambda_2 y, \lambda_3 r^{-1}(y) + \lambda_4 b^*)^\wedge(e) \\
= \; & (\lambda_1 r(\mathrm{ad}^*(x) b^*) + \lambda_2 r(\mathrm{ad}^*(x) r^{-1}(y)) + \lambda_1 \mathrm{ad}^*(a^*) r(b^*) + \lambda_2 \mathrm{ad}^*(a^*) y, \lambda_3 \mathrm{ad}^*(r(a^*)) r^{-1}(y) + \\
& \lambda_4 \mathrm{ad}^*(r(a^*)) b^* + \lambda_3 \mathrm{ad}^*(x) r^{-1}(y) + \lambda_4 \mathrm{ad}^*(x) b^*), \\
& N_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(\nabla_{(x,a^*)^\wedge}(y, b^*)^\wedge(e)) \\
& = N_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(r(\mathrm{ad}^*(x) r^{-1}(y)) + \mathrm{ad}^*(a^*) y, \mathrm{ad}^*(r(a^*)) b^* + \mathrm{ad}^*(x) b^*) \\
& = (\lambda_1 r(\mathrm{ad}^*(r(a^*)) b^*) + \lambda_1 r(\mathrm{ad}^*(x) b^*) + \lambda_2 r(\mathrm{ad}^*(x) r^{-1}(y)) + \lambda_2 \mathrm{ad}^*(a^*) y, \lambda_3 \mathrm{ad}^*(x) r^{-1}(y) + \\
& \quad \lambda_3 r^{-1}(\mathrm{ad}^*(a^*) y) + \lambda_4 \mathrm{ad}^*(r(a^*)) b^* + \lambda_4 \mathrm{ad}^*(x) b^*).
\end{aligned}$$

Therefore by Eq. (58), we get

$$\nabla_{(x,a^*)^\wedge} N_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(y, b^*)^\wedge(e) = N_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(\nabla_{(x,a^*)^\wedge}(y, b^*)^\wedge(e)).$$

Thus, $N_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}$ is parallel with respect to ∇, as needed. □

Proposition 4.9. *Let $(\mathfrak{g}, r)$ be a triangular Lie bialgebra. Let $\mathfrak{B}_p$ be the bilinear form on $\mathcal{D}(g) = \mathfrak{g}\oplus\mathfrak{g}^*$ given by Eq. (29) and let $\varphi : \mathcal{D}(\mathfrak{g}) = \mathfrak{g}\oplus\mathfrak{g}^* \to \mathcal{D}(\mathfrak{g})^* = \mathfrak{g}\oplus\mathfrak{g}^*$ be the linear map induced by $\mathfrak{B}_p$ through Eq. (8) for $\mathfrak{B} = \mathfrak{B}_p$. Define a family of linear endomorphisms of $\mathcal{D}(\mathfrak{g})$ by*

$$R_\mu(x, a^*) \equiv (\mu r(a^*) + x, -a^*), \quad \forall x \in \mathfrak{g}, a^* \in \mathfrak{g}^*, \mu \in \mathbb{R}.$$

Define $\tilde{r}_{\pm,\mu} \equiv R_\mu\varphi^{-1} \pm \varphi^{-1}$ and regard $\tilde{r}_{\pm,\mu}$ as elements of $\mathcal{D}(\mathfrak{g}) \otimes \mathcal{D}(\mathfrak{g})$. Then $(\mathcal{D}(\mathfrak{g}), \tilde{r}_{\pm,\mu})$ are factorizable quasitriangular Lie bialgebras.

Proof. First we prove that, for any $\mu \in \mathbb{R}$, R_μ is an extended $\mathcal{O}$-operator with extention id : $\mathcal{D}(\mathfrak{g}) \to \mathcal{D}(\mathfrak{g})$ of mass -1, that is, it satisfies Eq. (27) on $\mathcal{D}(\mathfrak{g})$. Recall the Lie algebra structure of $\mathcal{D}(\mathfrak{g})$ is given by Eq. (30). Then, for any $x, y \in \mathfrak{g}, a^*, b^* \in \mathfrak{g}^*$, we have

$$\begin{aligned}
& [R_\mu(x, a^*), R_\mu(y, b^*)]_{\mathcal{D}(\mathfrak{g})} = [(\mu r(a^*) + x, -a^*), (\mu r(b^*) + y, -b^*)]_{\mathcal{D}(\mathfrak{g})} \\
= \; & ([\mu r(a^*) + x, \mu r(b^*) + y] + \mathrm{ad}^*(-a^*)(\mu r(b^*) + y) - \mathrm{ad}^*(-b^*)(\mu r(a^*) + x), [a^*, b^*]_\delta - \\
& \mathrm{ad}^*(\mu r(a^*) + x)b^* + \mathrm{ad}^*(\mu r(b^*) + y)a^*).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
[(x, a^*), (y, b^*)]_{\mathcal{D}(\mathfrak{g})} &= ([x, y] + \mathrm{ad}^*(a^*)y - \mathrm{ad}^*(b^*)x, [a^*, b^*]_\delta + \mathrm{ad}^*(x)b^* - \mathrm{ad}^*(y)a^*) \\
R_\mu([R_\mu(x, a^*), (y, b^*)]_{\mathcal{D}(\mathfrak{g})}) &= (-\mu r([a^*, b^*]_\delta) + \mu^2 r(\mathrm{ad}^*(r(a^*))b^*) + \mu r(\mathrm{ad}^*(x)b^*) + \mu r(\mathrm{ad}^*(y)a^*) \\
&\quad +\mu[r(a^*), y] + [x, y] - \mathrm{ad}^*(a^*)y - \mu\mathrm{ad}^*(b^*)r(a^*) - \mathrm{ad}^*(b^*)x, \\
&\quad [a^*, b^*]_\delta - \mu\mathrm{ad}^*(r(a^*))b^* - \mathrm{ad}^*(x)b^* - \mathrm{ad}^*(y)a^*) \\
R_\mu([(x, a^*), R_\mu(y, b^*)]_{\mathcal{D}(\mathfrak{g})}) &= (-\mu r([a^*, b^*]_\delta) - \mu r(\mathrm{ad}^*(x)b^*) - \mu^2 r(\mathrm{ad}^*(r(b^*))a^*) - \mu r(\mathrm{ad}^*(y)a^*) \\
&\quad +\mu[x, r(b^*)] + [x, y] + \mu\mathrm{ad}^*(a^*)(r(b^*)) + \mathrm{ad}^*(a^*)y + \mathrm{ad}^*(b^*)x, \\
&\quad [a^*, b^*]_\delta + \mathrm{ad}^*(x)b^* + \mu\mathrm{ad}^*(r(b^*))a^* + \mathrm{ad}^*(y)a^*).
\end{aligned}$$

Therefore, by the fact that r is a homomorphism of Lie algebras (see Corollary 3.10), we get

$$[R_\mu(x, a^*), R_\mu(y, b^*)]_{\mathcal{D}(\mathfrak{g})} + [(x, a^*), (y, b^*)]_{\mathcal{D}(\mathfrak{g})} = R_\mu([R_\mu(x, a^*), (y, b^*)]_{\mathcal{D}(\mathfrak{g})}) + R_\mu([(x, a^*), R_\mu(y, b^*)]_{\mathcal{D}(\mathfrak{g})}).$$

On the other hand, from the proof of Lemma 4.7, we know that R_μ is skew-adjoint with respect to the nondegenerate symmetric invariant bilinear form $\mathfrak{B}_p$. So the conclusion follows from Corollary 4.4.(i) by setting $\mathfrak{g} = \mathcal{D}(\mathfrak{g})$, $R = R_\mu$, $\beta = \mathrm{id}_{\mathcal{D}(\mathfrak{g})}$ and $\mathfrak{B} = \mathfrak{B}_p$. □

Note that when $\mu = 0$, then Proposition 4.9 gives a special case of the famous "Drinfeld's double construction" [28] (in the original construction there is no restriction that $\mathfrak{g}$ is triangular, or even coboundary).

Proposition 4.10. *Let $(\mathfrak{g}, r)$ be a triangular Lie bialgebra such that r regarded as a linear map from $\mathfrak{g}^*$ to $\mathfrak{g}$ is invertible. Define two families of linear endomorphisms on $\mathcal{D}(\mathfrak{g})$ by*

$$\begin{aligned}
& N_\mu(x, a^*) = (x, \mu r^{-1}(x) - a^*), \quad \mu \in \mathbb{R}; \\
& N_{\kappa_1,\kappa_2}(x, a^*) = (\kappa_1 r(a^*) + \kappa_2 x, \frac{1-\kappa_2^2}{\kappa_1} r^{-1}(x) - \kappa_2 a^*), \quad \kappa_1, \kappa_2 \in \mathbb{R}, \kappa_2^2 \neq 1, \kappa_1 \neq 0,
\end{aligned}$$

for any $x \in \mathfrak{g}, a^* \in \mathfrak{g}^*$. Let $\varphi : \mathcal{D}(\mathfrak{g}) = \mathfrak{g}\oplus\mathfrak{g}^* \to \mathcal{D}(\mathfrak{g})^* = \mathfrak{g}\oplus\mathfrak{g}^*$ be the linear map induced by the bilinear form $\mathfrak{B}_p$ given by Eq. (29) through Eq. (8) for $\mathfrak{B} = \mathfrak{B}_p$. Define $\tilde{N}_{\pm,\mu} \equiv N_\mu\varphi^{-1} \pm \varphi^{-1}$, $\tilde{N}_{\pm,\kappa_1,\kappa_2} \equiv N_{\kappa_1,\kappa_2}\varphi^{-1} \pm \varphi^{-1}$ and regard $\tilde{N}_{\pm,\mu}$ and $\tilde{N}_{\pm,\kappa_1,\kappa_2}$ as elements of $\mathcal{D}(\mathfrak{g})\otimes\mathcal{D}(\mathfrak{g})$. Then $(\mathcal{D}(\mathfrak{g}), \tilde{N}_{\pm,\mu})$ and $(\mathcal{D}(\mathfrak{g}), \tilde{N}_{\pm,\kappa_1,\kappa_2})$ are factorizable quasitriangular Lie bialgebras.

Proof. In fact, according to Lemma 4.8, N_μ and N_{κ_1,κ_2} satisfy Eq. (24) on $\mathcal{D}(\mathfrak{g})$. Moreover, it is straightforward to check that $N_\mu^2 = \mathrm{id}$ and $N_{\kappa_1,\kappa_2}^2 = \mathrm{id}$. So both of them satisfy Eq. (27) on $\mathcal{D}(\mathfrak{g})$. On the other hand, by Lemma 4.7, they are skew-adjoint with respect to the nondegenerate

symmetric invariant bilinear form $\mathfrak{B}_p$. So the conclusion follows from Corollary 4.4.(i) by setting $\mathfrak{g} = \mathcal{D}(\mathfrak{g})$, $R = N_\mu$ or N_{κ_1,κ_2}, $\beta = \mathrm{id}_{\mathcal{D}(\mathfrak{g})}$ and $\mathfrak{B} = \mathfrak{B}_p$. □

4.3. **Factorizable type II quasitriangular Lie bialgebras.** We now consider the "factorizable" case of type II quasitriangular Lie bialgebras.

Definition 4.11. A type II quasitriangular Lie bialgebra $(\mathfrak{g}, r)$ is called **factorizable** if the symmetric part β of r regarded as a linear map from $\mathfrak{g}^*$ to $\mathfrak{g}$ is invertible.

The following conclusion is the type II analogue of the "factorizable" property of quasitriangular Lie bialgebras [39].

Proposition 4.12. *Let $(\mathfrak{g}, r)$ be a factorizable type II quasitriangular Lie bialgebra. Put $\tilde{r} = \alpha + i\beta : \mathfrak{g} \oplus i\mathfrak{g} \to \mathfrak{g} \oplus i\mathfrak{g}$, where α and β are defined by Eq. (35). Then any element $x \in \mathfrak{g}$ admits a unique decomposition:*

$$x = x_+ + x_-,$$

with $(x_+, x_-) \in \mathrm{Im}(\tilde{r} \oplus \tilde{r}^t) \subset \mathfrak{g} \oplus i\mathfrak{g}$, where $\tilde{r}$ and $\tilde{r}^t$ are restricted to linear maps from $i\mathfrak{g}^* \subset \mathfrak{g} \oplus i\mathfrak{g}$ to $\mathfrak{g} \oplus i\mathfrak{g}$.

Proof. Since $\tilde{r} + \tilde{r}^t = 2i\beta$ and $\beta : \mathfrak{g}^* \to \mathfrak{g}$ is invertible, we have

$$x = \tilde{r}(\frac{\beta^{-1}(x)}{2i}) + \tilde{r}^t(\frac{\beta^{-1}(x)}{2i}) \in \mathrm{Im}(\tilde{r} \oplus \tilde{r}^t) \subset \mathfrak{g} \oplus i\mathfrak{g}, \quad \forall x \in \mathfrak{g}.$$

On the other hand, if there exist $a^*, b^* \in \mathfrak{g}^*$ such that $x = \tilde{r}(ia^*) + \tilde{r}^t(ia^*) = \tilde{r}(ib^*) + \tilde{r}^t(ib^*)$. Then $0 = \tilde{r}(ia^* - ib^*) + \tilde{r}^t(ia^* - ib^*) = -2\beta(a^* - b^*)$. Since $\beta : \mathfrak{g}^* \to \mathfrak{g}$ is invertible, we obtain $a^* = b^*$. So the conclusion follows. □

The following result provides a class of factorizable type II quasitriangular Lie bialgebras (hence a new class of (coboundary) Lie bialgebras).

Proposition 4.13. *Let $(\mathfrak{g}, r)$ be a triangular Lie bialgebra such that r regarded as a linear map from $\mathfrak{g}^*$ to $\mathfrak{g}$ is invertible. Let $\mathfrak{B}_p$ be the bilinear form on $\mathcal{D}(g) = \mathfrak{g} \oplus \mathfrak{g}^*$ given by Eq. (29) and let $\varphi : \mathcal{D}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^* \to \mathcal{D}(\mathfrak{g})^* = \mathfrak{g} \oplus \mathfrak{g}^*$ be the linear map induced by $\mathfrak{B}_p$ through Eq. (8) for $\mathfrak{B} = \mathfrak{B}_p$. Define a family of linear endomorphisms on $\mathcal{D}(\mathfrak{g})$ by*

$$J_{\lambda,\mu}(x, a^*) = (\lambda r(a^*) + \mu x, \frac{-1 - \mu^2}{\lambda} r^{-1}(x) - \mu a^*), \quad \lambda, \mu \in \mathbb{R}, \lambda \neq 0.$$

Set $\tilde{r}_{\pm,\lambda,\mu} \equiv J_{\lambda,\mu}\varphi^{-1} \pm \varphi^{-1}$ and regard $\tilde{r}_{\pm,\lambda,\mu}$ as elements of $\mathcal{D}(\mathfrak{g}) \otimes \mathcal{D}(\mathfrak{g})$. Then $(\mathcal{D}(\mathfrak{g}), \tilde{r}_{\pm,\lambda,\mu})$ are factorizable type II quasitriangular Lie bialgebras.

Proof. In fact, according to Lemma 4.8, for any $\lambda, \mu \in \mathbb{R}$, $J_{\lambda,\mu}$ satisfies Eq. (24) on $\mathcal{D}(\mathfrak{g})$. Moreover, it is straightforward to check that $J_{\lambda,\mu}^2 = -\mathrm{id}$. So $J_{\lambda,\mu}$ satisfy Eq. (26) for $\kappa = 1$ on $\mathcal{D}(\mathfrak{g})$. On the other hand, by Lemma 4.7, $J_{\lambda,\mu}$ is skew-adjoint with respect to the nondegenerate symmetric invariant bilinear form $\mathfrak{B}_p$. So the conclusion follows from Corollary 4.4.(ii) by setting $\mathfrak{g} = \mathcal{D}(\mathfrak{g})$, $R = J_{\lambda,\mu}$, $\beta = \mathrm{id}_{\mathcal{D}(\mathfrak{g})}$ and $\mathfrak{B} = \mathfrak{B}_p$. □

Remark 4.14.
(i) A linear transformation on a Lie algebra $\mathfrak{g}$ satisfying Eq. (24) and $J^2 = -\mathrm{id}$ is called a **complex structure** on $\mathfrak{g}$. Suppose a Lie algebra is self-dual with respect to a nondegenerate symmetric invariant bilinear form. According to Corollary 4.4.(ii), a complex structure on this Lie algebra that is self adjoint with respect to the bilinear form gives rise to a coboundary Lie bialgebra structure on this Lie algebra. This idea was pursued further in [34] in the study of Poisson-Lie groups.
(ii) The complex structure $J_{-1,0}$ has already been found in [16].

5. $\mathcal{O}$-operators, PostLie algebras and dendriform trialgebras

In this section, we reveal a PostLie algebra structure underneath the $\mathcal{O}$-operators. We then show that there is a close relationship between PostLie algebras and dendriform trialgebras of Loday and Ronco [36] in parallel to the relationship [12] between Pre-Lie algebras and dendriform bialgebras.

5.1. $\mathcal{O}$-operators and PostLie algebras.

We begin with recalling the concept of a PostLie algebra from an operad study [51].

Definition 5.1. ([51]) A **(left) PostLie algebra** is a $\mathbb{R}$-vector space L with two bilinear operations $\circ$ and $[,]$ which satisfy the relations:

$$[x,y] = -[y,x], \tag{59}$$

$$[[x,y],z] + [[z,x],y] + [[y,z],x] = 0, \tag{60}$$

$$z\circ(y\circ x) - y\circ(z\circ x) + (y\circ z)\circ x - (z\circ y)\circ x + [y,z]\circ x = 0, \tag{61}$$

$$z\circ[x,y] - [z\circ x, y] - [x, z\circ y] = 0, \tag{62}$$

for all $x,y \in L$. Eq. (59) and Eq. (60) mean that L is a Lie algebra for the bracket $[,]$, and we denote it by $(\mathfrak{G}(L),[,])$. Moreover, we say that $(L,[,],\circ)$ **is a PostLie algebra structure on** $(\mathfrak{G}(L),[,])$. On the other hand, it is straightforward to check that L is also a Lie algebra for the operation:

$$\{x,y\} \equiv x\circ y - y\circ x + [x,y], \quad \forall x,y\in L. \tag{63}$$

We shall denote it by $(\mathcal{G}(L),\{,\})$ and say that $(\mathcal{G}(L),\{,\})$ **has a compatible PostLie algebra structure given by** $(L,[,],\circ)$. A **homomorphism between two PostLie algebras** is defined as a linear map between the two PostLie algebras that preserves the corresponding operations.

Remark 5.2.
(i) The notion of PostLie algebra was introduced in [51] (in its "right version"), where it is pointed out that **PostLie**, the operad of PostLie algebras, is the Koszul dual of **ComTrias**, the operad of **commutative trialgebras**.
(ii) If the bracket $[,]$ in the definition of PostLie algebra happens to be trivial, then a PostLie algebra is a **pre-Lie algebra** [11].

Lemma 5.3. *Let $(L,[,],\circ)$ be a PostLie algebra. Define $\rho : L\to \mathfrak{gl}(L)$ by $\rho(x)y = x\circ y$ for any $x,y\in L$. Then $(\mathfrak{G}(L),[\ ,\ ],\rho)$ is a $(\mathcal{G}(L),\{\ ,\ \})$-Lie algebra.*

Proof. By Eq. (61), ρ is a representation of $(\mathcal{G}(L),\{\ ,\ \})$. Then by Eq. (62), ρ is a Lie algebra homomorphism from $(\mathcal{G}(L),\{,\})$ to $\mathrm{Der}_{\mathbb{R}}(\mathfrak{G}(L))$. □

Theorem 5.4. *Let $\mathfrak{g}$ be a Lie algebra and $(\mathfrak{k},\pi)$ be a $\mathfrak{g}$-Lie algebra. Let $r:\mathfrak{k}\to\mathfrak{g}$ be an $\mathcal{O}$-operator of weight λ.*

(i) *The following operations define a PostLie algebra structure on the underlying vector space of $\mathfrak{k}$:*

$$[x,y] \equiv \lambda[x,y]_{\mathfrak{k}}, \quad x\circ y \equiv r(x)\cdot y, \quad x,y\in\mathfrak{k}, \tag{64}$$

where $[,]_{\mathfrak{k}}$ is the original Lie bracket of $\mathfrak{k}$.

(ii) *r is a Lie algebra homomorphism from $\mathcal{G}(\mathfrak{k})$ to $\mathfrak{g}$, where $\mathfrak{k}$ is taken as a PostLie algebra with the operations $([,],\circ)$ defined in Eq. (64).*

(iii) *If* Ker(r) *is an ideal of* $(\mathfrak{k}, [,]_{\mathfrak{k}})$, *then there exists an induced PostLie algebra structure on* $r(\mathfrak{k})$ *given by*

$$[r(x), r(y)]_r \equiv \lambda r([x,y]_{\mathfrak{k}}), \quad r(x) \circ_r r(y) \equiv r(r(x)\cdot y), \quad \forall x, y \in \mathfrak{k}. \tag{65}$$

Further, r is a homomorphism of PostLie algebras.

Proof. (i) Since $\mathfrak{k}$ is a Lie algebra, Eq. (59) and Eq. (60) obviously hold. Furthermore, for any $x, y, z \in \mathfrak{k}$, we have

$$\begin{aligned}
& z\circ(y\circ x) - y\circ(z\circ x) + (y\circ z)\circ x - (z\circ y)\circ x + [y,z]\circ x \\
= \;& r(z)\cdot(r(y)\cdot x) - r(y)\cdot(r(z)\cdot x) + r(r(y)\cdot z)\cdot x - r(r(z)\cdot y)\cdot x + \lambda r([y,z]_{\mathfrak{k}})\cdot x \\
= \;& ([r(z), r(y)]_{\mathfrak{g}} - r(r(z)\cdot y - r(y)\cdot z + \lambda[z,y]_{\mathfrak{k}}))\cdot x = 0
\end{aligned}$$

So Eq. (61) holds. Similarly, Eq. (62) holds, too.

(ii) By Definition 5.1, for any $x, y \in \mathfrak{k}$ we have

$$r(\{x,y\}) = r(x\circ y - y\circ x + [x,y]) = r(r(x)\cdot y - r(y)\cdot x + \lambda[x,y]_{\mathfrak{k}}) = [r(x), r(y)]_{\mathfrak{g}}.$$

(iii) We first prove that the multiplications given by Eq. (65) are well-defined. In fact, let $x_1, y_1, x_2, y_2 \in \mathfrak{k}$ such that $r(x_1) = r(x_2)$ and $r(y_1) = r(y_2)$. Since $x_1 - x_2, y_1 - y_2 \in$ Ker(r) and Ker(r) is an ideal of $(\mathfrak{k}, [,]_{\mathfrak{k}})$, we have

$$\begin{aligned}
r(x_1)\circ_r r(y_1) &= r(r(x_1)\cdot y_1) = r(r(x_2 + (x_1 - x_2))\cdot(y_2 + (y_1 - y_2))) \\
&= r(r(x_2)\cdot y_2 + r(x_2)\cdot(y_1 - y_2)) \\
&= r(r(x_2)\cdot y_2) + [r(x_2), r(y_1 - y_2)]_{\mathfrak{g}} + r(r(y_1 - y_2)\cdot x_2) - \lambda r([x_2, y_1 - y_2]_{\mathfrak{k}}) \\
&= r(r(x_2)\cdot y_2) = r(x_2)\circ_r r(y_2).
\end{aligned}$$

Also, $[r(x_1), r(y_1)]_r = [r(x_2) + r(x_1 - x_2), r(y_1) + r(y_1 - y_2)]_r = [r(x_2), r(y_2)]_r$. Furthermore, we have $r([x,y]) = [r(x), r(y)]_r$ and $r(x\circ y) = r(x)\circ_r r(y)$ for any $x, y \in \mathfrak{k}$. Thus, $(r(\mathfrak{k}), [,]_r, \circ_r)$ is a PostLie algebra since applying r to the PostLie algebra axioms of $(\mathfrak{k}, [,], \circ)$ gives the PostLie algebra axioms of $(r(\mathfrak{k}), [,]_r, \circ_r)$. Finally, the last statement in Item (iii) is clear. ◻

Corollary 5.5. *Let* $\mathfrak{g}$ *be a Lie algebra. Then there is a compatible PostLie algebra structure on* $\mathfrak{g}$ *if and only if there exists a* $\mathfrak{g}$*-Lie algebra* $(\mathfrak{k}, \pi)$ *and an invertible* $\mathcal{O}$*-operator* $r: \mathfrak{k}\to\mathfrak{g}$ *of weight* 1.

Proof. Suppose that $\mathfrak{g}$ has a compatible PostLie algebra structure given by $(L, [,], \circ)$, that is, $\mathcal{G}(L) = \mathfrak{g}$. By Lemma 5.3, $(\mathfrak{G}(L), \rho, [,])$ is a $\mathfrak{g}$-Lie algebra, where $\rho: L\to\mathfrak{gl}(L)$ is defined as $\rho(x)y = x\circ y$ for any $x, y \in L$. Moreover, the equation $\{x,y\} = x\circ y - y\circ x + [x,y]$ means that $\mathrm{id}: \mathfrak{G}(L)\to\mathcal{G}(L) = \mathfrak{g}$ is an $\mathcal{O}$-operator of weight 1. Furthermore, id is obviously invertible.

Conversely, suppose that $(\mathfrak{k}, \pi)$ is a $\mathfrak{g}$-Lie algebra and $r: \mathfrak{k}\to\mathfrak{g}$ is an invertible $\mathcal{O}$-operator weight 1. Since Ker$(r) = \{0\}$, by Theorem 5.4, there is a PostLie algebra structure on $r(\mathfrak{k}) = \mathfrak{g}$ given by Eq. (65) for $\lambda = 1$. Moreover, it is obvious that $(r(\mathfrak{k}) = \mathfrak{g}, [,]_r, \circ_r)$ (for $\lambda = 1$) is a compatible PostLie algebra structure on $(\mathfrak{g}, [,]_{\mathfrak{g}})$. ◻

Corollary 5.6. *Let* $\mathfrak{g}$ *be a Lie algebra and* $R: \mathfrak{g}\to\mathfrak{g}$ *be a Rota-Baxter operator of weight* $\lambda\in\mathbb{R}$, *that is, it satisfies Eq.* (14). *Then there is a PostLie algebra structure on* $\mathfrak{g}$ *given by*

$$[x,y] \equiv \lambda[x,y]_{\mathfrak{g}}, \quad x\circ y \equiv [R(x), y]_{\mathfrak{g}}, \quad \forall x, y \in \mathfrak{g}. \tag{66}$$

If in addition, R is invertible, then there is a compatible PostLie algebra structure on $\mathfrak{g}$ *given by*

$$[x,y] \equiv \lambda R([R^{-1}(x), R^{-1}(y)]_{\mathfrak{g}}), \quad x\circ y \equiv R([x, R^{-1}(y)]_{\mathfrak{g}}), \quad \forall x, y \in \mathfrak{g}.$$

Proof. The conclusion follows from Theorem 5.4. ◻

We next give examples of PostLie algebras by applying Corollary 5.6.

Example 5.7. Let $\mathfrak{g}$ be a complex simple Lie algebra, $\mathfrak{h}$ be its Cartan subalgebra, Δ be its root system and $\Delta_+ \subset \Delta$ be the set of positive roots (with respect to some fixed order). For $\alpha \in \Delta$, let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ be the corresponding root space. Put $\mathfrak{n}_\pm = \oplus_{\alpha\in\Delta_+}\mathfrak{g}_{\pm\alpha}$, $\mathfrak{b}_\pm = \mathfrak{h} + \mathfrak{n}_\pm$. Then we have $\mathfrak{g} = \mathfrak{b}_+ + \mathfrak{n}_-$ as decomposition of two subalgebras. Let $P_{\mathfrak{b}_+} : \mathfrak{g} \to \mathfrak{b}_+ \hookrightarrow \mathfrak{g}$ and $P_{\mathfrak{n}_-} : \mathfrak{g} \to \mathfrak{n}_- \hookrightarrow \mathfrak{g}$ be the projections onto the subalgebras $\mathfrak{b}_+$ and $\mathfrak{n}_-$ respectively. Then $-P_{\mathfrak{b}_+}$ and $-P_{\mathfrak{n}_-}$ are Rota-Baxter operators of weight 1. Define new operations on $\mathfrak{g}$ as follows:

$$[x,y] \equiv [x,y]_{\mathfrak{g}}, \quad x \circ_{\mathfrak{b}_+} y \equiv -[P_{\mathfrak{b}_+}(x), y]_{\mathfrak{g}}, \quad \forall x,y \in \mathfrak{g}. \tag{67}$$

By Corollary 5.6, $([,], \circ_{\mathfrak{b}_+})$ defines a PostLie algebra structure on $\mathfrak{g}$. If

$$\{H_i\}_{i=1,\dots,n} \cup \{X_\alpha\}_{\alpha\in\Delta_+} \cup \{X_{-\alpha}\}_{\alpha\in\Delta_+}$$

is a basis of $\mathfrak{g}$, then the PostLie operations defined by Eq. (67) can be computed as follows:

$$\begin{gathered}[x,y] = [x,y]_{\mathfrak{g}}, \quad X_{-\alpha} \circ_{\mathfrak{b}_+} y = 0, \quad H_i \circ_{\mathfrak{b}_+} H_j = 0, \quad H_i \circ_{\mathfrak{b}_+} X_\beta = -\langle \beta, \alpha_i\rangle X_\beta,\\ X_\alpha \circ_{\mathfrak{b}_+} H_i = \langle \alpha, \alpha_i\rangle X_\alpha, \quad X_\alpha \circ_{\mathfrak{b}_+} X_\beta = -N_{\alpha,\beta}X_{\alpha+\beta}, \quad \forall x,y \in \mathfrak{g},\ \alpha \in \Delta_+, \beta \in \Delta.\end{gathered}$$

Similarly, with the same bracket $[\,,\,]$ and with $x \circ_{\mathfrak{n}_-} y \equiv -[P_{\mathfrak{n}_-}(x), y]_{\mathfrak{g}}$, we obtain another PostLie algebra structure $([,], \circ_{\mathfrak{n}_-})$ on $\mathfrak{g}$.

The following result is interesting considering the importance of Baxter Lie algebra in integrable systems [10, 43].

Corollary 5.8. *Let $(\mathfrak{g}, R)$ be a Baxter Lie algebra, that is, $R : \mathfrak{g} \to \mathfrak{g}$ satisfies Eq. (27). Define the following operations on the underlying vector space of $\mathfrak{g}$ by*

$$[x,y] \equiv [x,y]_{\mathfrak{g}}, \quad x \circ_\pm y \equiv \Big[\Big(\frac{R \pm 1}{\mp 2}\Big)(x), y\Big]_{\mathfrak{g}}, \quad \forall x,y \in \mathfrak{g}.$$

Then $(\mathfrak{g}, [,], \circ_\pm)$ are PostLie algebras.

Proof. From the discussion at the end of Section 2.3, we show that $(R \pm 1)/(\mp 2)$ both are Rota-Baxter operators of weight 1. So the conclusion follows from Corollary 5.6. ◻

By Corollary 3.10 and Theorem 5.4, we also obtain the following close relation between quasitriangular Lie bialgebras and PostLie algebras.

Corollary 5.9. *Let $(\mathfrak{g}, r)$ be a quasitriangular Lie bialgebra. Define $\beta \in \mathfrak{g}\otimes\mathfrak{g}$ by Eq. (35). Then*

$$[a^*, b^*] \equiv -2\mathrm{ad}^*(\beta(a^*))b^*, \quad a^* \circ b^* \equiv \mathrm{ad}^*(r(a^*))b^*, \quad \forall a^*, b^* \in \mathfrak{g}^*,$$

defines a PostLie algebra structure on $\mathfrak{g}^*$. If in addition, r regarded as a linear map from $\mathfrak{g}^*$ to $\mathfrak{g}$ is invertible, then the following operations define a compatible PostLie algebra structure on $\mathfrak{g}$:

$$[x,y] \equiv -2r(\mathrm{ad}^*(\beta(r^{-1}(x)))r^{-1}(y)), \quad x \circ y \equiv r(\mathrm{ad}^*(x)r^{-1}(y)), \quad \forall x,y \in \mathfrak{g}.$$

It is obvious that for any Lie algebra $(\mathfrak{g}, [,])$, $(\mathfrak{g}, [,], -[,])$ is a PostLie algebra. Moreover, we have the following conclusion.

Theorem 5.10. *Let $(\mathfrak{g}, [,])$ be a semisimple Lie algebra. Then any PostLie algebra structure $(\mathfrak{g}, [,], \circ)$ (on $\mathfrak{g}, [,]$)) is given by*

$$x \circ y = [f(x), y], \quad \forall x,y \in \mathfrak{g},$$

where $f : \mathfrak{g} \to \mathfrak{g}$ is a Rota-Baxter operator of weight 1.

Proof. Let $L_\circ$ be the left multiplication operator with respect to $\circ$, that is, $L_\circ(x)y = x \circ y$ for any $x, y \in \mathfrak{g}$. Then by Eq. (62), $L_\circ$ is a derivation of the Lie algebra $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, every derivation of $\mathfrak{g}$ is inner. Therefore, there exists a linear map $f : \mathfrak{g} \to \mathfrak{g}$ such that

$$x \circ y = L_\circ(x)y = \mathrm{ad} f(x)y = [f(x), y], \quad \forall x, y \in \mathfrak{g}.$$

Moreover, by Eq. (61), we see that

$$[[f(y), f(z)], x] = [f([f(y), z] + [y, f(z)] + [y, z]), x], \quad \forall x, y, z \in \mathfrak{g}.$$

Since the center of $\mathfrak{g}$ is zero, f is a Rota-Baxter operator of weight 1. □

Remark 5.11. In fact, the above conclusion can be extended to a Lie algebra $\mathfrak{g}$ satisfying that the center of $\mathfrak{g}$ is zero and every derivation of $\mathfrak{g}$ is inner (such a Lie algebra is called **complete** [37]). On the other hand, note that f is a Rota-Baxter operator of weight 1 if and only if $R = 2f + 1$ is an extended $\mathcal{O}$-operator with extention $\mathrm{id} : \mathfrak{g} \to \mathfrak{g}$ of mass -1, i.e., R satisfies Eq. (27). In particular, the classification of the linear maps satisfy Eq. (27) for every complex semisimple Lie algebra was given in [43].

5.2. **Dendriform trialgebras and PostLie algebras: a commutative diagram.** Dendriform dialgebras [35] and trialgebras [36] are introduced with motivation from algebraic K-theory and topology. Dendriform dialgebras are known to give pre-Lie algebras. We will show that a more general correspondence holds between dendriform trialgebras and PostLie algebras.

Definition 5.12. ([36]) A **dendriform trialgebra** $(A, \prec, \succ, \cdot)$ is a vector space A equipped with three bilinear operations $\{\prec, \succ, \cdot\}$ satisfying the following equations:

$$(x \prec y) \prec z = x \prec (y \star z), \quad (x \succ y) \prec z = x \succ (y \prec z),$$

$$(x \star y) \succ z = x \succ (y \succ z), \quad (x \succ y) \cdot z = x \succ (y \cdot z),$$

$$(x \prec y) \cdot z = x \cdot (y \succ z), \quad (x \cdot y) \prec z = x \cdot (y \prec z), \quad (x \cdot y) \cdot z = x \cdot (y \cdot z),$$

for $x, y, z \in A$. Here $\star \equiv \prec + \succ + \cdot$.

According to [36], the product given by $x \star y = x \prec y + x \succ y + x \cdot y$ defines an associative product on A. Moreover, if the operation $\cdot$ is trivial, then a dendriform trialgebra is a **dendriform dialgebra** [35].

Proposition 5.13. *Let $(A, \prec, \succ, \cdot)$ be a dendriform trialgebra. Then the products*

$$[x, y] \equiv x \cdot y - y \cdot x, \quad x \circ y \equiv x \succ y - y \prec x, \quad \forall x, y \in A, \tag{68}$$

make $(A, [\,,\,], \circ)$ into a PostLie algebra.

Proof. We will only prove Axiom (62). The other axioms are similarly proved. For any $x, y, z \in A$, we have

$$\begin{aligned}
& z \circ [x, y] - [z \circ x, y] - [x, z \circ y] \\
= \ & z \succ (x \cdot y - y \cdot x) - (x \cdot y - y \cdot x) \prec z - (z \succ x - x \prec z) \cdot y + y \cdot (z \succ x - x \prec z) - \\
& x \cdot (z \succ y - y \prec z) + (z \succ y - y \prec z) \cdot x \\
= \ & z \succ (x \cdot y) - (z \succ x) \cdot y - z \succ (y \cdot x) + (z \succ y) \cdot x - (x \cdot y) \prec z + x \cdot (y \prec z) + \\
& (y \cdot x) \prec z - y \cdot (x \prec z) + (x \prec z) \cdot y - x \cdot (z \succ y) + y \cdot (z \succ x) - (y \prec z) \cdot x = 0.
\end{aligned}$$

□

It is easy to see that Eq. (63) and Eq. (68) fit into the commutative diagram:

$$\begin{tikzcd}
\text{Dendriform trialgebra} \arrow[r, "x\prec y+x\succ y+x\cdot y"] \arrow[d, "{[x,y]=x\cdot y-y\cdot x}"', "x\circ y=x\succ y-y\prec x"] & \text{Associative algebra} \arrow[d, "x\star y-y\star x"] \\
\text{PostLie algebra} \arrow[r, "{x\circ y-y\circ x+[x,y]}"] & \text{Lie algebra}
\end{tikzcd}$$

When the operation $\cdot$ of the dendriform trialgebra and the bracket $[,]$ of the PostLie algebra are trivial, we obtain the following commutative diagram introduced in [12] (see also [2, 3]):

$$\begin{tikzcd}
\text{Dendriform dialgebra} \arrow[r, "x\prec y+x\succ y"] \arrow[d, "x\succ y-y\prec x"] & \text{Associative algebra} \arrow[d, "x\star y-y\star x"] \\
\text{Pre-Lie algebra} \arrow[r, "x\circ y-y\circ x"] & \text{Lie algebra}
\end{tikzcd}$$

6. Triple Lie algebras and examples of non-abelian generalized Lax pairs

Our primary goal in this section is to apply our study of PostLie algebras in Section 5 to study integrable systems. To construct non-abelian generalized Lax pairs, we formulate the setup of a triple Lie algebra that is consistent with the classical r-matrix approach to integrable systems [13, 28, 43]. We then show that new situations where this setup applies are provided by PostLie algebras from Rota-Baxter operators on complex simple Lie algebras.

6.1. **Triple Lie algebra and a typical example of non-abelian generalized Lax pairs.** We introduce the following concept to obtain self-dual nonabelian generalized Lax pairs.

Definition 6.1. A **triple Lie algebra** consists of the following data $(\mathfrak{g},[\ ,\ ]_0,\rho,[\ ,\ ],\mathfrak{B},r,\lambda)$ where

(i) $(\mathfrak{g},[\ ,\ ]_0)$ is a Lie algebra;

(ii) $[\ ,\ ]$ is another Lie bracket on the underlying vector space of $\mathfrak{g}$ and $\rho:\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$ is a representation of $(\mathfrak{g},[\ ,\ ]_0)$ such that $(\mathfrak{g},[\ ,\ ],\rho)$ is a $(\mathfrak{g},[,]_0)$-Lie algebra. Denote $x\cdot y\equiv\rho(x)y$, for any $x,y\in\mathfrak{g}$;

(iii) $\mathfrak{B}:\mathfrak{g}\otimes\mathfrak{g}\to\mathbb{R}$ is a nondegenerate symmetric bilinear form such that Eq. (3) and Eq. (4) hold for $(\mathfrak{a},[,]_{\mathfrak{a}})=(\mathfrak{g},[,])$.

(iv) r is in $\mathfrak{g}\otimes\mathfrak{g}$ such that the corresponding linear map $r:\mathfrak{g}^*\to\mathfrak{g}$ through Eq. (34) has the property that the following bilinear operation defines a Lie bracket on $\mathfrak{g}$:

$$[x,y]_r\equiv\tilde{r}(x)\cdot y-\tilde{r}(y)\cdot x+\lambda[x,y],\quad\forall x,y\in\mathfrak{g},\tag{69}$$

for certain $\lambda\in\mathbb{R}$ and for $\tilde{r}\equiv r\varphi:\mathfrak{g}\to\mathfrak{g}$ where φ is defined by Eq. (8).

A triple Lie algebra is so named because of the three Lie algebra structures $[\ ,\ ]_0$, $[\ ,\ ]$ and $[\ ,\ ]_r$ on the same underlying vector space $\mathfrak{g}$. It often happens that the invariant condition in Eq. (3) implies Eq. (4), so Eq. (3) is enough in a triple Lie algebra. This is the case in the following classical example. This is also the case of PostLie algebras considered in Section 6.2.

Example 6.2. An example of triple Lie algebra is the following well-known setting considered by Semonov-Tian-Shansky [13, 28, 43] in integrable systems. Let $(\mathfrak{g},[\ ,\ ]_0)$ be a semisimple Lie algebra. Let $\rho=\operatorname{ad}$ be the adjoint representation. Let $(\mathfrak{g},[\ ,\ ])$ be $(\mathfrak{g},[\ ,\ ]_0)$ and let $\mathfrak{B}(\ ,\ )$ be its

Killing form. Let r be a skew-symmetric solution of the **generalized classical Yang-Baxter equation (GCYBE)**:

$$(\mathrm{ad}(x)\otimes\mathrm{id}\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{ad}(x)\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{ad}(x))([r_{12},r_{13}]+[r_{12},r_{23}]+[r_{13},r_{23}])=0,\quad \forall x\in\mathfrak{g}.$$

Then Eq. (69) with $\lambda=0$ defines a Lie bracket on the underlying vector space of $\mathfrak{g}$.

Remark 6.3. (i) Let G be a simply connected Lie group whose Lie algebra is $\mathfrak{g}$. Then any representation $\rho:\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$ is determined by a left invariant flat connection ∇ on G through

$$\rho(x)y\equiv\nabla_{\hat{x}}\hat{y}(e),\quad \forall x,y\in\mathfrak{g}.$$

Here $\hat{x},\hat{y}$ are the left invariant vector fields generated by $x,y\in\mathfrak{g}$ and e is the identity element of G. Moreover, a bilinear form $\mathfrak{B}$ satisfying Eq. (4) for $(\mathfrak{a},[,]_{\mathfrak{a}})=(\mathfrak{g},[,])$ corresponds to a left invariant pseudo-Riemannian metric which is compatible with the connection ∇ [38].

(ii) By the study in Section 2, an obvious ansatz satisfies condition (iv) in Definition 6.1 is that $\tilde{r}$ is an extended $\mathcal{O}$-operator of weight λ with extention β of mass (ν,κ,μ) for $\nu\neq 0$.

For a triple Lie algebra, there exists a **Lie-Poisson structure** [50] on $\mathfrak{g}^*$, defined by

$$\{f,g\}_r(a^*)\equiv\langle[df(a^*),dg(a^*)]_r,a^*\rangle,\quad \forall f,g\in C^\infty(\mathfrak{g}^*),a^*\in\mathfrak{g}^*. \tag{70}$$

Proposition 6.4. *Given a triple Lie algebra* $(\mathfrak{g},[\ ,\ ]_0,\rho,[\ ,\ ],\mathfrak{B},r,\lambda)$ *in Definition 6.1, any two smooth functions on* $\mathfrak{g}^*$ *that are invariant under the dual representation of* ρ *and the coadjoint representation of* $(\mathfrak{g},[\ ,\ ])$ *are in involution with respect to the Lie-Poisson structure.*

Proof. If f and g are two smooth functions on $\mathfrak{g}^*$ that are invariant under the dual representation of ρ and the coadjoint representation of $\mathfrak{g}$, then

$$\begin{aligned}\{f,g\}_r(a^*) &= \langle\rho(\tilde{r}(df(a^*)))dg(a^*),a^*\rangle-\langle\rho(\tilde{r}(dg(a^*)))df(a^*),a^*\rangle+\lambda\langle[df(a^*),dg(a^*)],a^*\rangle\\ &= -\langle dg(a^*),\rho^*(\tilde{r}(df(a^*)))a^*\rangle+\langle df(a^*),\rho^*(\tilde{r}(dg(a^*)))a^*\rangle+\lambda\langle df(a^*),\mathrm{ad}^*(dg(a^*))a^*\rangle\\ &= 0,\end{aligned}$$

as needed. □

The above proposition motivates us to consider Hamiltonian systems on $\mathfrak{g}^*$ with the Lie-Poisson structure $\{,\}_r$.

Theorem 6.5. *Let a triple Lie algebra* $(\mathfrak{g},[\ ,\ ]_0,\rho,[\ ,\ ],\mathfrak{B},r,\lambda)$ *be given. Let* $\mathcal{H}$ *(the Hamiltonian) be a smooth function on* $\mathfrak{g}^*$ *which is invariant under the dual representation of* ρ *and the coadjoint representation of* $(\mathfrak{g},[\ ,\ ])$. *Let* $\{e_i\}_{1\le i\le\dim\mathfrak{g}}$ *be a basis of* $\mathfrak{g}$ *with dual basis* $\{e^i\}_{1\le i\le\dim\mathfrak{g}}$ *with respect to* $\mathfrak{B}$. *Let*

$$\Omega\equiv\sum_i e_i\otimes e^i\in\mathfrak{g}\otimes\mathfrak{g} \tag{71}$$

be the Casimir element. Let $L,M:\mathfrak{g}^*\to\mathfrak{g}$ *be smooth maps defined by* $L(a^*)=(a^*\otimes 1)(\Omega)$ *and* $M(a^*)=\tilde{r}(d\mathcal{H}(a^*))$, $a^*\in\mathfrak{g}^*$. *Then* $(\mathfrak{g},\rho,\mathfrak{g},L,M)$ *is a self-dual nonabelian generalized Lax pair for the Hamiltonian system* $(\mathfrak{g}^*,\{\ ,\ \}_r,\mathcal{H})$ *in the sense of Definition 2.2.*

Proof. For any $f\in C^\infty(\mathfrak{g}^*)$, we have

$$\begin{aligned}\frac{d}{dt}f(a^*) &= \{\mathcal{H},f\}_r\\ &= \langle\rho(\tilde{r}(d\mathcal{H}(a^*)))df(a^*),a^*\rangle-\langle\rho(\tilde{r}(df(a^*)))d\mathcal{H}(a^*),a^*\rangle+\lambda\langle[d\mathcal{H}(a^*),df(a^*)],a^*\rangle\\ &= -\langle df(a^*),\rho^*(\tilde{r}(d\mathcal{H}(a^*)))a^*\rangle,\quad \forall a^*\in\mathfrak{g}^*.\end{aligned}$$

Since $\mathfrak{B}$ satisfies Eq. (4) for $(\mathfrak{a}, [,]_{\mathfrak{a}}) = (\mathfrak{g}, [,])$, it is easy to show that (cf. Lemma 4.2)

$$(\rho(x) \otimes \mathrm{id} + \mathrm{id} \otimes \rho(x))\Omega = 0, \quad \forall x, y \in \mathfrak{g}. \tag{72}$$

Then

$$\frac{d}{dt}L(a^*) = -((\rho^*(\tilde{r}(d\mathcal{H}(a^*)))a^*) \otimes \mathrm{id})(\Omega) = (a^* \otimes 1)((\rho(M(a^*)) \otimes \mathrm{id})\Omega) = -(a^* \otimes 1)((\mathrm{id} \otimes \rho(M(a^*)))\Omega). \tag{73}$$

Hence

$$\frac{d}{dt}L(a^*) = -\rho(M(a^*))((a^* \otimes 1)(\Omega)) = -\rho(M(a^*))L(a^*).$$

Therefore $(\mathfrak{g}, \rho, \mathfrak{g}, L, M)$ is a self-dual nonabelian generalized Lax pair. □

The invariant condition under the dual representation of ρ holds automatically in some interesting cases, such as in Example 6.2 and Section 6.2. This is also true for Corollary 6.8.

Remark 6.6. Consider the triple Lie algebra in Example 6.2 and take $\mathcal{H}$ to be a smooth function on $\mathfrak{g}^*$ which is invariant under the coadjoint representation of $(\mathfrak{g}, [\ ,\ ])$. Applying Theorem 6.5, we have

$$\frac{d}{dt}L(a^*) = [L(a^*), M(a^*)], \quad \forall a^* \in \mathfrak{g}^*,$$

that is, (L, M) is a **Lax pair** in the ordinary sense [13].

We next show that $(\mathfrak{g}, \rho, \mathfrak{g}, L, M)$ admits certain "nonabelian generalized r-matrix ansatz". First, the Poisson bracket of smooth functions on $\mathfrak{g}^*$ defined by Eq. (70) can be extended to $\mathfrak{g}$-valued functions in an obvious way: with the notations as above, let E and F be two $\mathfrak{g}$-valued smooth functions on $\mathfrak{g}$ such that

$$E = \sum_s E_s e^s, \quad F = \sum_s F_s e^s,$$

where $E_s, F_s \in C^\infty(\mathfrak{g}^*)$, then

$$\{E, F\}_r = \sum_{s,t} \{E_s, F_t\}_r e^s \otimes e^t.$$

Suppose that r is skew-symmetric (resp. symmetric) and

$$r = \sum_{s,t} a_{st} e_s \otimes e^t = -\sum_{s,t} a_{ts} e^s \otimes e_t \ \ (\text{resp. } r = \sum_{s,t} a_{st} e_s \otimes e^t = \sum_{s,t} a_{ts} e^s \otimes e_t).$$

Then $\tilde{r}(e_s) = r(\varphi(e_s)) = -\sum_t a_{ts} e_t$ (resp. $\tilde{r}(e_s) = r(\varphi(e_s)) = \sum_t a_{ts} e_t$). Set $[e_s, e_t] = \sum_k d^k_{st} e_k$, $[e_s, e^t] = \sum_k \tilde{d}^k_{st} e^k$ and $e_l \cdot e^s = \sum_t c^t_{ls} e^t$. Since $L(a^*) = \sum_s L_s(a^*) e^s$, where $L_s(a^*) = \langle e_s, a^* \rangle$, we have

$$\begin{aligned}
\{L, L\}_r(a^*) &= \sum_{s,t} \{L_s, L_t\}_r(a^*) e^s \otimes e^t = \sum_{s,t} \langle [dL_s(a^*), dL_t(a^*)]_r, a^* \rangle e^s \otimes e^t \\
&= \sum_{s,t} \langle [e_s, e_t]_r, a^* \rangle e^s \otimes e^t = \sum_{s,t} \langle \tilde{r}(e_s) \cdot e_t - \tilde{r}(e_t) \cdot e_s + \lambda [e_s, e_t], a^* \rangle e^s \otimes e^t \\
&= \sum_{s,t,l} \langle -a_{ls} e_l \cdot e_t + a_{lt} e_l \cdot e_s, a^* \rangle e^s \otimes e^t + \lambda \sum_{s,t,k} d^k_{st} \langle e_k, a^* \rangle e^s \otimes e^t.
\end{aligned}$$

$$(\text{resp. } \{L, L\}_r(a^*) = \sum_{s,t,l} \langle a_{ls} e_l \cdot e_t - a_{lt} e_l \cdot e_s, a^* \rangle e^s \otimes e^t + \lambda \sum_{s,t,k} d^k_{st} \langle e_k, a^* \rangle e^s \otimes e^t)$$

However, by Eq. (72) we have

$$\sum_s e_l \cdot e_s \otimes e^s = -\sum_s e_s \otimes e_l \cdot e^s.$$

Letting $a^* \otimes 1$ act on both sides of the above equation, we see that

$$\sum_s \langle a^*, e_l \cdot e_s\rangle e^s = -\sum_s \langle a^*, e_s\rangle e_l \cdot e^s.$$

Therefore

$$\begin{aligned}\langle -a_{ls} e_l \cdot e_t, a^*\rangle e^s \otimes e^t &= \langle a_{ls} e_t, a^*\rangle e^s \otimes e_l \cdot e^t,\\ \langle a_{lt} e_l \cdot e_s, a^*\rangle e^s \otimes e^t &= -\langle a_{lt} e_s, a^*\rangle e_l \cdot e^s \otimes e_t.\end{aligned}$$

Furthermore, since $\mathfrak{B}([e_s, e_t], e^k) = -\mathfrak{B}(e_t, [e_s, e^k])$, we have $d^k_{st} = -\tilde{d}^t_{sk}$. In conclusion, we obtain the "nonabelian generalized r-matrix ansatz" that we are looking for (Eq. (7)).

Theorem 6.7. *When r is skew-symmetric (resp. symmetric), the self-dual nonabelian generalized Lax pair in Theorem 6.5 satisfies*

$$\{L, L\}_r = \sum_{s,t,l,k} \{a_{ls} c^t_{lk} \langle e_k, a^*\rangle - a_{lt} c^s_{lk} \langle e_k, a^*\rangle - \lambda \tilde{d}^t_{sk} \langle e_k, a^*\rangle\} e^s \otimes e^t.$$

$$(\textit{resp. } \{L, L\}_r = \sum_{s,t,l,k} \{-a_{ls} c^t_{lk} \langle e_k, a^*\rangle + a_{lt} c^s_{lk} \langle e_k, a^*\rangle - \lambda \tilde{d}^t_{sk} \langle e_k, a^*\rangle\} e^s \otimes e^t)$$

Thus by Proposition 2.5, we have

Corollary 6.8. *With the conditions in Theorem 6.7, for any two smooth functions f and g on $\mathfrak{g}$ that are invariant under the representation ρ and the adjoint representation of $(\mathfrak{g}, [\ ,\ ])$, we have $\{f \circ L, g \circ L\}_r = 0$.*

6.2. **The case of PostLie algebras.** We now apply Rota-Baxter operators and PostLie algebras to give an example of triple Lie algebra.

Theorem 6.9. *Let $(\mathfrak{g}, [,]_{\mathfrak{g}})$ be a complex simple Lie algebra. Let $R : \mathfrak{g} \to \mathfrak{g}$ be a Rota-Baxter operator of weight 1. Let $([\ ,\ ], \circ)$ denote the PostLie algebra structure on $\mathfrak{g}$ given by Eq. (66) for $\lambda = 1$. Let $(\mathfrak{g}, \rho, [\ ,\ ])$ denote the $(\mathfrak{g}, \{,\})$-Lie algebra given by Lemma 5.3. Let $\mathfrak{B}$ denote the Killing form on $\mathfrak{g}$. Suppose there exists an $r \in \mathfrak{g} \otimes \mathfrak{g}$ such that*

$$[x, y]_r \equiv \rho(\tilde{r}(x))y - \rho(\tilde{r}(y))x + \tilde{\lambda}[x, y] = [R(\tilde{r}(x)), y]_{\mathfrak{g}} + [x, R(\tilde{r}(y))]_{\mathfrak{g}} + \tilde{\lambda}[x, y]_{\mathfrak{g}}, \quad \forall x, y \in \mathfrak{g}, \tag{74}$$

defines a Lie bracket on the underlying vector space of $\mathfrak{g}$, where $\tilde{\lambda} \in \mathbb{R}$ and $\tilde{r} \equiv r\varphi : \mathfrak{g} \to \mathfrak{g}$ and φ is defined by Eq. (8). Then

(i) *$(\mathfrak{g}, \{\ ,\ \}, \rho, [\ ,\ ], \mathfrak{B}, r, \tilde{\lambda})$ is a triple Lie algebra.*
(ii) *Let $\mathcal{H}$ (the Hamiltonian) be a smooth function on $\mathfrak{g}^*$ which is invariant under the coadjoint representation of $(\mathfrak{g}, [\ ,\ ])$. Let Ω be the Casimir element in Eq. (71). Let $L, M : \mathfrak{g}^* \to \mathfrak{g}$ be smooth maps defined by $L(a^*) = (a^* \otimes 1)(\Omega)$ and $M(a^*) = \tilde{r}(d\mathcal{H}(a^*))$, $a^* \in \mathfrak{g}^*$. Then $(\mathfrak{g}, \rho, \mathfrak{g}, L, M)$ is a self-dual nonabelian generalized Lax pair for the Hamiltonian system $(\mathfrak{g}^*, \{\ ,\ \}_r, \mathcal{H})$ where $\{\ ,\ \}_r$ is the Lie-Poisson structure defined in Eq. (70).*
(iii) *If r is symmetric or skew-symmetric, then for any two smooth functions f and g on $\mathfrak{g}$ that are invariant under the adjoint representation of $(\mathfrak{g}, [\ ,\ ])$, we have $\{f \circ L, g \circ L\}_r = 0$.*

Proof. (i) Since $\mathfrak{B}$ is the Killing form, it satisfies Eq. (3) for $(\mathfrak{a}, [,]_{\mathfrak{a}}) = (\mathfrak{g}, [,])$. Moreover, we have

$$\mathfrak{B}([R(x), y], z) + \mathfrak{B}(y, [R(x), z]) = 0 \Leftrightarrow \mathfrak{B}(\rho(x)y, z) + \mathfrak{B}(y, \rho(x)z) = 0, \quad \forall x, y, z \in \mathfrak{g},$$

that is, $\mathfrak{B}$ also satisfies Eq. (4) for $(\mathfrak{a}, [,]_{\mathfrak{a}}) = (\mathfrak{g}, [,])$.

(ii) If $\mathcal{H}$ is a smooth function which is invariant under the coadjoint action of G, then $\mathcal{H}$ is also invariant under the dual representation of ρ since for any $x \in \mathfrak{g}$, $a^* \in \mathfrak{g}^*$,

$$\langle d\mathcal{H}(a^*), \rho^*(x)(a^*)\rangle = -\langle [R(x), d\mathcal{H}(a^*)], a^*\rangle = \langle d\mathcal{H}(a^*), \mathrm{ad}^*(R(x))a^*\rangle = 0.$$

By Theorem 6.5, $(\mathfrak{g}, \rho, \mathfrak{g}, L, M)$ is a self-dual nonabelian generalized Lax pair.

(iii) In this case, f and g are also invariant under the representation ρ since by definition $\rho(x)y = [R(x), y]$, for any $x, y \in \mathfrak{g}$. Then the conclusion follows from Corollary 6.8. □

Appendix: Extended $\mathcal{O}$-operators and affine geometry on Lie groups

In this appendix, motivated by [10], we provide a geometric explanation of the extended $\mathcal{O}$-operators. Let K be a simply connected Lie group whose Lie algebra is $\mathfrak{k}$. Let ∇ be a left invariant connection on K, which, according to [27], is specified by a linear map $\tilde{r} : \mathfrak{k} \to \mathfrak{gl}(\mathfrak{k})$ through

$$\tilde{r}(x) \cdot y \equiv \nabla_{\hat{x}}\hat{y}(e), \quad \forall x, y \in \mathfrak{k},$$

where $\hat{x}, \hat{y}$ are the left invariant vector fields generated by $x, y \in \mathfrak{k}$ respectively and e is the identity element of K. Define a linear map $r : \mathfrak{k} \to \mathfrak{gl}(\mathfrak{k})$ by

$$r(x) \cdot y \equiv \nabla_{\hat{x}}\hat{y}(e) - \frac{\lambda}{2}[x, y]_{\mathfrak{k}} = \tilde{r}(x) \cdot y - \frac{\lambda}{2}[x, y]_{\mathfrak{k}}, \quad \forall x, y \in \mathfrak{k}.$$

Let $\mathfrak{g}$ be the Lie subalgebra of $\mathfrak{gl}(\mathfrak{k})$ generated by all $r(x)$. Then r is a linear map from $\mathfrak{k}$ to $\mathfrak{g}$. Furthermore, for any $x, y \in \mathfrak{k}$, we have

$$\begin{aligned}
[x, y]_R &\equiv r(x) \cdot y - r(y) \cdot x + \lambda[x, y]_{\mathfrak{k}} \\
&= \tilde{r}(x) \cdot y - \frac{\lambda}{2}[x, y]_{\mathfrak{k}} - \tilde{r}(y) \cdot x + \frac{\lambda}{2}[y, x]_{\mathfrak{k}} + \lambda[x, y]_{\mathfrak{k}} \\
&= \tilde{r}(x) \cdot y - \tilde{r}(y) \cdot x = \nabla_{\hat{x}}\hat{y}(e) - \nabla_{\hat{y}}\hat{x}(e).
\end{aligned}$$

So if $[,]_R$ defines a Lie bracket on the underlying vector space of $\mathfrak{k}$ and K_R denotes the corresponding simply connected Lie group, then the left invariant connection determined by

$$\nabla_{\hat{x}}\hat{y}(e) = r(x) \cdot y + \frac{\lambda}{2}[x, y]_{\mathfrak{k}}$$

is torsion-free, where $x, y \in \mathfrak{k}$ and e is the identity element of K_R. Now we assume that $\mathfrak{k}$ is a $\mathfrak{g}$-Lie algebra, that is, the image of r belongs to $\mathrm{Der}_{\mathbb{R}}(\mathfrak{k})$, the Lie subalgebra consisting of the derivations of $\mathfrak{k}$. This is equivalent to

$$\nabla_{\hat{x}}([y, z]_{\mathfrak{k}})^{\wedge}(e) = [\nabla_{\hat{x}}\hat{y}(e), z]_{\mathfrak{k}} + [y, \nabla_{\hat{x}}\hat{z}(e)]_{\mathfrak{k}}, \quad \forall x, y, z \in \mathfrak{k}.$$

Next we compute the curvature tensor $R(,)$ of ∇:

$$\begin{aligned}
R(\hat{x}, \hat{y})\hat{z}(e) &= (\nabla_{\hat{x}}\nabla_{\hat{y}} - \nabla_{\hat{y}}\nabla_{\hat{x}} - \nabla_{[x,y]_R^{\wedge}})\hat{z}(e) \\
&= r(x) \cdot (r(y) \cdot z) + \frac{\lambda}{2}[x, r(y) \cdot z]_{\mathfrak{k}} + \frac{\lambda}{2}r(x) \cdot [y, z]_{\mathfrak{k}} + \frac{\lambda^2}{4}[x, [y, z]_{\mathfrak{k}}]_{\mathfrak{k}} - r(y) \cdot (r(x) \cdot z) \\
&\quad -\frac{\lambda}{2}r(y) \cdot [x, z]_{\mathfrak{k}} - \frac{\lambda}{2}[y, r(x) \cdot z]_{\mathfrak{k}} - \frac{\lambda^2}{4}[y, [x, z]_{\mathfrak{k}}]_{\mathfrak{k}} - r([x, y]_R) \cdot z - \frac{\lambda}{2}[r(x) \cdot y, z]_{\mathfrak{k}} \\
&\quad +\frac{\lambda}{2}[r(y) \cdot x, z]_{\mathfrak{k}} - \frac{\lambda^2}{2}[[x, y]_{\mathfrak{k}}, z]_{\mathfrak{k}} \\
&= ([r(x), r(y)]_{\mathfrak{g}} - r([x, y]_R)) \cdot z - \frac{\lambda^2}{4}[[x, y]_{\mathfrak{k}}, z]_{\mathfrak{k}},
\end{aligned}$$

where the Lie bracket $[,]_{\mathfrak{g}}$ on $\mathfrak{g}$ is the commutator bracket of linear transformations. Since $[,]_{\mathfrak{k}}$ satisfies the Jacobi identity, we can re-interpret the "Jacobi identity condition" in Proposition 2.9.(ii) as the **first Bianchi's identity** for the curvature tensor of a torsion-free connection.

Theorem. *With the same notations as above, suppose that* $\mathfrak{k}$ *is a* $\mathfrak{g}$*-Lie algebra and* $[,]_R$ *defines a Lie bracket on the underlying vector space of* $\mathfrak{k}$*. Denote* K_R *for the corresponding simply connected Lie group. Let* $\beta : \mathfrak{k} \to \mathfrak{g}$ *be a linear map such that* β *is* $\mathfrak{g}$*-invariant of mass* κ *and also of mass* μ*, i.e., the following equations hold*

$$\kappa\beta(\xi \cdot x) = \kappa[\xi, \beta(x)]_{\mathfrak{g}}, \quad \mu\beta(\xi \cdot x) = \mu[\xi, \beta(x)]_{\mathfrak{g}}, \quad \forall \xi \in \mathfrak{g}, x \in \mathfrak{k}.$$

Let r *and* β *satisfy Eq.* (12)*. Then the corresponding curvature tensor (of the left invariant torsion-free connection* ∇*)*

$$R_e(x,y)z \equiv \kappa[\beta(x), \beta(y)]_{\mathfrak{g}} \cdot z + \mu\beta([x,y]_{\mathfrak{k}}) \cdot z - \frac{\lambda^2}{4}[[x,y]_{\mathfrak{k}}, z]_{\mathfrak{k}}, \quad \forall x, y, z \in \mathfrak{k},$$

is $\mathfrak{g}$*-invariant, that is,*

$$\xi \cdot R_e(x,y)z - R_e(x,y)\xi \cdot z - R_e(\xi \cdot x, y)z - R_e(x, \xi \cdot y)z = 0, \quad \forall x, y, z \in \mathfrak{k}, \xi \in \mathfrak{g}.$$

In particular, setting $\xi = r(w)$*,* $w \in \mathfrak{k}$*, then the curvature tensor is covariantly constant which in turn is equivalent to the Lie group* K_R *being an affine locally symmetric space.*

Proof. The first statement depends on a direct computation. Moreover, combining with the fact that ∇ is torsion-free, we see that K_R is affine locally symmetric (cf. [27]). □

Remark. The above conclusion is a generalization of Theorem 3.7 in [10].

References

[1] M. Adler, On a trace functional for formal pseudodifferential operators and the symplectic structures for Korteweg-de Vries type equations, *Invent. Math.* **50** (1979) 219-248.

[2] M. Aguiar, Pre-Poisson algebras, *Lett. Math. Phys.* **54** (2000) 263-277.

[3] M. Aguiar, Infinitesimal bialgebras, pre-Lie and dendriform algebras, in “ Hopf algebras”, *Lecture Notes in Pure and Applied Mathematics* **237** (2004) 1-33.

[4] O. Babelon and C.-M. Viallet, Integrable models, Yang-Baxter equations, and quantum groups. part I. *Ref. S.I.S.S.A.* **54** EP (may 89) (preprint Trieste).

[5] O. Babelon and C.-M. Viallet, Hamiltonian structures and Lax equations, *Phys. Lett. B* **237** (1990) 411-416.

[6] C. Bai, L. Guo and X. Ni, $\mathcal{O}$-operators on associative algebras and associative Yang-Baxter equations, preprint (2009).

[7] G. Baxter, An analytic problem whose solution follows from a simple algebraic identity, *Pacific J. Math.* **10** (1960) 731-742.

[8] A. A. Belavin and V. G. Drinfeld, Solutions of the classical Yang-Baxter equation for simple Lie algebras. *Funct. Anal. Appl.* **16** (1982) 159-180, **17** (1983) 220-221.

[9] A. A. Belavin and V. G. Drinfeld, Triangle equations and simple Lie algebras, *Classical Reviews in Mathematics and Mathematical Physics* **1**, Harwood Academic Publishers, Amsterdam (1998).

[10] M. Bordemann, Generalized Lax pairs, the modified classical Yang-Baxter equation, and affine geometry of Lie groups, *Comm. Math. Phys.* **135** (1990) 201-216.

[11] D. Burde, Left-symmetric algebras, or pre-Lie algebras in geometry and physics, *Cent. Eur. J. Math.* **4** (2006) 323-357.

[12] F. Chapoton, Un theéorème de Cartier-Milnor-Moore-Quillen pour les bigébres dendriformes et les algébres braces, *J. Pure Appl. Algebra* **168** (2002) 1-18.

[13] V. Chari and A. Pressley, A guide to quantum groups, Cambridge University Press, Cambridge (1994).

[14] A. Connes and D. Kreimer, Hopf algebras, renormalization and noncommutative geometry, *Comm. Math. Phys.* **199** (1998) 203-242.

[15] A. Connes and D. Kreimer, Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem., *Comm. Math. Phys.*, **210** (2000) 249-273.

[16] A. Diatta and A. Medina, Classical Yang-Baxter equation and left invariant affine geometry on Lie groups, *Manuscripta Math.* **114** (2004) 477-486.

[17] V. G. Drinfeld, Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equations, *Soviet Math. Dokl.* **27** (1983) 68-71.

[18] V. G. Drinfeld, On Poisson homogenous spaces of Poisson-Lie groups, *Theoret. Math. Phys.* **95** (1993) 524-525.
[19] K. Ebrahimi-Fard, Loday-type algebras and the Rota-Baxter relation, *Lett. Math. Phys.* **61** (2002) 139-147.
[20] K. Ebrahimi-Fard and L. Guo, Rota-Baxter algebras in renormalization of perturbative quantum field theory, in “Universality and Renormalization”, *Fields Institute Communicatins* **50**, Amer. Math. Soc. (2007) 47-105.
[21] L. D. Faddeev, Integrable models in 1 + 1-dimensional quantum field theory, in: Recent Advances in Field Theory and Statistical Mechanics, Les Houches, Elsevier Science Publishers: Amsterdam, (1984) 563-608.
[22] J. M. Figueroa-O’Farrrill and S. Stanciu, On the structure of symmetric self-dual Lie algebras, *J. Math. Phys.* **37** (1996) 4121-4134.
[23] L. Guo, Algebraic Birkhoff decomposition and its applications, in “Automorphic Forms and the Langlands Program”, International Press, (2008) 283-323.
[24] T. J. Hodge and T. Levasseur, Primitive ideals of $\mathbb{C}_q[SL(3)]$, *Comm. Math. Phys.* **156** (1993) 581-605.
[25] T. J. Hodge and M. Yakimov, The double and dual of a quasitriangular Lie bialgebra, *Math. Res. Lett.* **8** (2001) 91-105.
[26] A. W. Knapp, Lie groups beyond an introduction, Birkhäuser, Berlin (1996).
[27] S. Kobayashi and K. Nomizu, Foundations of differential geometry. New York: Wiley, Vol. I (1963) and Vol. II (1969).
[28] Y. Kosmann-Schwarzbach, Lie bialgebras, Poisson Lie groups and dressing transformations, in “Integrability of nonlinear systems”, *Lecture Notes in Physics* **495**, Springer, Berlin (1997) 104-170.
[29] Y. Kosmann-Schwarzbach and F. Magri, Poisson-Lie groups and complete integrability, I. Drinfeld bialgebras, dual extensions and their canonical representations, *Ann. Inst. Henri Poincaré, Phys. Théor. A* **49** (1988) 433-460.
[30] B. Kostant, The solution to a generalized Toda lattice and representation theory, *Adv. Math.* **34** (1979) 195-338.
[31] B. A. Kupershmidt, What a classical r-matrix really is, *J. Nonlinear Math. Phy.* **6** (1999) 448-488.
[32] P. D. Lax, Integrals of nonlinear equations of evolution and solitary waves, *Commun. Pure Appl. Math.* **21** (1968) 467-490.
[33] S. L. Levendorskii and Y. S. Soibelman, Algebras of functions on compact quantum groups, Schubert cells, and quantum tori, *Commun. Math. Phys.* **139** (1991) 141-170.
[34] Z.-J. Liu and M. Qian, Generalized Yang-Baxter equations, Koszul operators and Poisson-Lie groups, *J. Diff. Geom.* **35** (1992) 399-414.
[35] J.-L. Loday, Dialgebras, in “ Dialgebras and related operads”, *Lecture Notes in Math.* **1763**, Springer, Berlin (2001) 7-66.
[36] J.-L. Loday and M. Ronco, Trialgebras and families of polytopes, in “Homotopy Theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic K-theory”, *Comtep. Math.* **346** (2004) 369-398.
[37] D. J. Meng, Some results on complete Lie algebras, *Comm. Algebra* **22** (1994) 5457-5507.
[38] J. Milnor, Curvatures of left invariant metrics on Lie groups, *Adv. Math.* **21** (1976) 293-329.
[39] N. Reshetikhin and M. Semenov-Tian-Shansky, Quantum R-matrices and factorization problems, *J. Geom. Phys.* **5** (1998) 533-550.
[40] A. Reyman and M. Semenov-Tian-Shansky, Compatible Poisson structures for Lax equations: an r-matrix approach, *Phys. Lett. A* **130** (1988) 456-460.
[41] A. Reyman and M. Semenov-Tian-Shansky, Group-theoretical methods in the theory of finite-dimensional integrable systems, in: Integrable Systems II. Dynamical Systems VII, *Encyclopaedia of Math. Sciences*, Springer-Verlag, Berlin, **16** (1994), 116-220.
[42] G.-C. Rota, Baxter operators, an introduction, in ”Gian-Carlo Rota on Combinatorics: Introductory papers and commentaries” (Joseph P.S. Kung, Ed.), Birkhäuser, Boston (1995) 504-512.
[43] M. Semonov-Tian-Shansky, What is a classical R-matrix? *Funct. Anal. Appl.* **17** (1983) 259-272.
[44] M. Semonov-Tian-Shansky, Dressing transformations and Poisson group actions, *Publ. RIMS, Kyoto Univ.* **21** (1985) 1237-1260.
[45] E. K. Sklyanin, On complete integrability of the Landau-Lifschitz equation. Preprint Leningr. Otd. Mat. Inst. E-3-79 (1979).
[46] E. K. Sklyanin, The quantum inverse scattering method. Zap. Nauch. Sem. LOMI **95** (1980) 55-128.
[47] A. Stolin, Some remarks on Lie bialgebra structures on simple complex Lie algebras, *Comm. Algebra* **27** (1999) 4289-4302.

[48] W. Symes, Systems of Toda type, inverse spectral problems, and representation theory, *Invent. Math.* **59** (1980) 13-51.
[49] W. Symes, Hamiltonian group actions and integrable systems. *Physica D* **1** (1980) 339-374.
[50] I. Vaisman, Lecture on the geometry of Poisson manifolds, Progress in Mathematics **118**, Birkhäuser Verlag, Basel (1994).
[51] B. Vallette, Homology of generalized partition posets, *J. Pure Appl. Algebra* **208** (2007) 699-725.
[52] M. Yakimov, Symplectic leaves of complex reductive Poisson-Lie groups, *Duke Math. J.* **112** (2002) 453-509.

Chern Institute of Mathematics & LPMC, Nankai University, Tianjin 300071, P.R. China
E-mail address: xiangn_math@yahoo.cn

Chern Institute of Mathematics & LPMC, Nankai University, Tianjin 300071, P.R. China
E-mail address: baicm@nankai.edu.cn

Department of Mathematics and Computer Science, Rutgers University, Newark, NJ 07102
E-mail address: liguo@newark.rutgers.edu